Lagrangian quantum field theory in momentum picture

III. Free vector fields

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Contents

Abstract

Free vector fields, satisfying the Lorenz condition, are investigated in details in the momentum picture of motion in Lagrangian quantum field theory. The field equations are equivalently written in terms of creation and annihilation operators and on their base the commutation relations are derived. Some problems concerning the vacuum and state vectors of free vector field are discussed. Special attention is paid to peculiarities of the massless case; in particular, the electromagnetic field is explored. Several Lagrangians, describing free vector fields, are considered and the basic consequences of them are pointed and compared.

1. Introduction

This paper is devoted to an exploration of two types of free vector fields in the momentum picture of Lagrangian quantum field theory[1]: massive vector fields and massless vector fields, the latter satisfying the Lorenz[2] condition as addition to the Lagrangian formalism. Since the massive free vector fields satisfy the Lorenz condition as a consequence of the Euler-Lagrange equations, both kinds of fields are treated on almost equal footing in the present work. However, the massless case has its own peculiarities to which special attention is paid. Most of the known results, concerning the mentioned fields in Heisenberg picture, are reproduced in momentum picture of motion. New results are obtained too. For example, the field equations in terms of creation and annihilation operators and the (second) quantization of electromagnetic field in Lorenz gauge (imposed on the fields potentials as operators), a special case of which is the quantization in Coulomb gauge.

The work may be regarded as a continuation of [13, 14], where free scalar and spinor, respectively, fields are studied in momentum picture.

The basic moments of the method, we will follow in this work, are the following ones:

(i) In Heisenberg picture is fixed a (second) non-quantized and non-normally ordered operator-valued Lagrangian, which is supposed to be polynomial (or convergent power series) in the field operators and their first partial derivatives;

(ii) As conditions additional to the Lagrangian formalism are postulated the commutativity between the components of the momentum operator (see (2.6) below) and the Heisenberg relations between the field operators and momentum operator (see (2.7) below);

(iii) Following the Lagrangian formalism in momentum picture, the creation and annihilation operators are introduced and the dynamical variables and field equations are written in their terms;

(iv) From the last equations, by imposing some additional restrictions on the creation and annihilation operators, the (anti)commutation relations for these operators are derived;

(v) At last, the vacuum and normal ordering procedure are defined, by means of which the theory can be developed to a more or less complete form.

The main difference of the above scheme from the standard one is that we postulate the below-written relations (2.6) and (2.7) and, then, we look for compatible with them and the field equations (anti)commutation relations. (Recall, ordinary the (anti)commutation relations are postulated at first and the validity of the equations (2.6) and (2.7) is explored after that [15].)

The organization of the material is as follows.

In Sect. 2 are reviewed, for reference purposes, the basic aspects of momentum picture

[1] In this paper we considered only the Lagrangian (canonical) quantum field theory in which the quantum fields are represented as operators, called field operators, acting on some Hilbert space, which in general is unknown if interacting fields are studied. These operators are supposed to satisfy some equations of motion, from them are constructed conserved quantities satisfying conservation laws, etc. From the view-point of present-day quantum field theory, this approach is only a preliminary stage for more or less rigorous formulation of the theory in which the fields are represented via operator-valued distributions, a fact required even for description of free fields. Moreover, in non-perturbative directions, like constructive and conformal field theories, the main objects are the vacuum mean (expectation) values of the fields and from these are reconstructed the Hilbert space of states and the acting on it fields. Regardless of these facts, the Lagrangian (canonical) quantum field theory is an inherent component of the most of the ways of presentation of quantum field theory adopted explicitly or implicitly in books like [1–9]. Besides, the Lagrangian approach is a source of many ideas for other directions of research, like the axiomatic quantum field theory [3, 8, 9].

[2] The Lorenz condition and gauge (see below equation (3.6b)) are named in honor of the Danish theoretical physicist Ludwig Valentin Lorenz (1829–1891), who has first published it in 1867 [10] (see also [11, pp. 268-269, 291]); however this condition was first introduced in lectures by Bernhard G. W. Riemann in 1861 as pointed in [11, p. 291]. It should be noted that the *Lorenz* condition/gauge is quite often erroneously referred to as the Loren*tz* condition/gauge after the name of the Dutch theoretical physicist Hendrik Antoon Lorentz (1853–1928) as, e.g., in [3, p. 18] and in [12, p. 45].

of motion of quantum field theory. The description of free vector fields in this picture is presented in Sect. 3.

The structure of the solutions of the field equations is analyzed in Sect. 4. Decompositions of these solutions, equivalent to the Fourier decompositions in Heisenberg picture, are established. A suitably normalized system of classical solutions of the field equations is constructed. The creation and annihilation operators for the fields considered are introduced in Sect. 5 on a base of the decompositions and system of classical solutions just mentioned. A physical interpretation of these operators is derived from the Heisenberg relations, which are external to the Lagrangian formalism. At this point, the first problem with the massless case, concerning the angular momentum operator, appears. In Sect. 6, the operators of the dynamical variables of free vector fields (satisfying the Lorenz condition) are calculated in Heisenberg picture of motion in terms of creation and annihilation operators in momentum picture. Special attention is paid to the spin angular momentum operator and the above mentioned problem is analyzed further.

In Sect. 7, the field equations are equivalently rewritten in terms of creation and annihilation operators. As a consequence of them, the dynamical variables in momentum picture are found. It should be mentioned, in the massless case, the creation and annihilation operators corresponding to the degrees of freedom, 'parallel' to the 4-momentum variable, do not enter in the field equations. In Sect. 8, the commutation relations for free vector fields satisfying the Lorenz conditions are derived. They also do not include the just-mentioned operators. The commutators between the components of spin angular momentum operator and between them and the charge operator are calculate on the base of the established commutation relations. It is pointed that these relations play a role of field equations under the hypotheses they are derived. To the normal ordering procedure and definition of vacuum is devoted Sect. 9. Problems, regarding state vectors and physical interpretation of creation and annihilation operators in the Lagrangian formalism, are considered in Sect. 10.

Some peculiarities of the massless case are explored in Sect. 11. It is pointed that, generally, new suppositions are required for the treatment of creation and annihilation operators, connected with the degrees of freedom 'parallel' to the 4-momentum, which are the cause for the problems arising in the massless case. Two such hypotheses are analyzed. The obtained formalism is applied to a description of the electromagnetic field. In fact, it provides a new quantization of this field in which the Lorenz conditions is imposed directly on the field operators, which is completely different with respect to the one used in Gupta-Bleuler quantization. It is shown that for an electromagnetic field no problems arise, due to a suitable definition of the normal ordering procedure. The basic relation of quantum field theory of free electromagnetic field are written explicitly.

Sect. 12 contain a discussion of some Lagrangians suitable for description of free vector fields satisfying the Lorenz condition. The basic consequences of these Lagrangians are pointed and compared. As a 'best' Lagrangian is pointed the one which is charge-symmetric and, hence, in which the spin-statistics theorem is encoded. It is proved that the quantum field theories, arising from the considered Lagrangians, became identical after the normal ordering procedure is applied.

In Sect. 13 is analyzed the role of the Lorenz condition, when studying massless free vector fields. This is done by investigating a massless vector field with a Lagrangian equal to the one of a massive vector field with vanishing mass and without imposing the Lorenz condition as a subsidiary condition on the field operators. Sect. 14 closes the paper.

The books [1–3] will be used as standard reference works on quantum field theory. Of course, this is more or less a random selection between the great number of (text)books and papers on the theme to which the reader is referred for more details or other points of view. For this end, e.g., [4, 5, 16] or the literature cited in [1–5, 16] may be helpful.

Throughout this paper $\hbar$ denotes the Planck's constant (divided by 2π), c is the velocity of light in vacuum, and i stands for the imaginary unit. The superscripts $\dagger$ and $\top$ means respectively Hermitian conjugation and transposition of operators or matrices, the superscript $*$ denotes complex conjugation, and the symbol $\circ$ denotes compositions of mappings/operators.

By δ_{fg}, or δ_f^g or δ^{fg} ($:= 1$ for $f = g$, $:= 0$ for $f = g$) is denoted the Kronecker δ-symbol, depending on arguments f and g, and $\delta^n(y)$, $y \in \mathbb{R}^n$, stands for the n-dimensional Dirac δ-function; $\delta(y) := \delta^1(y)$ for $y \in \mathbb{R}$.

The Minkowski spacetime is denoted by M. The Greek indices run from 0 to $\dim M - 1 = 3$. All Greek indices will be raised and lowered by means of the standard 4-dimensional Lorentz metric tensor $\eta^{\mu\nu}$ and its inverse $\eta_{\mu\nu}$ with signature $(+\,-\,-\,-)$. The Latin indices $a, b, \dots$ run from 1 to $\dim M - 1 = 3$ and, usually, label the spacial components of some object. The Einstein's summation convention over indices repeated on different levels is assumed over the whole range of their values.

At the end, a technical remark is in order. The derivatives with respect to operator-valued (non-commuting) arguments will be calculated according to the rules of the classical analysis of commuting variables, which is an everywhere silently accepted practice [1, 15]. As it is demonstrated in [17], this is not quite correct but does not lead to incorrect results when free vector fields are concerned.

2. The momentum picture

In this section, we present a summary of the momentum picture in quantum field theory, introduce in [18] and developed in [19].

Let us consider a system of quantum fields, represented in Heisenberg picture of motion by field operators $\tilde{\varphi}_i(x)\colon \mathcal{F} \to \mathcal{F}$, $i = 1, \dots, n \in \mathbb{N}$, acting on the system's Hilbert space $\mathcal{F}$ of states and depending on a point x in Minkowski spacetime M. Here and henceforth, all quantities in Heisenberg picture will be marked by a tilde (wave) "$\tilde{\ }$" over their kernel symbols. Let $\tilde{\mathcal{P}}_\mu$ denotes the system's (canonical) momentum vectorial operator, defined via the energy-momentum tensorial operator $\tilde{\mathcal{T}}^{\mu\nu}$ of the system, viz.

$$\tilde{\mathcal{P}}_\mu := \frac{1}{c} \int\limits_{x^0=\text{const}} \tilde{\mathcal{T}}_{0\mu}(x)\, \mathrm{d}^3\boldsymbol{x}. \tag{2.1}$$

Since this operator is Hermitian, $\tilde{\mathcal{P}}_\mu^\dagger = \tilde{\mathcal{P}}_\mu$, the operator

$$\mathcal{U}(x, x_0) = \exp\Big(\frac{1}{\mathrm{i}\hbar} \sum_\mu (x^\mu - x_0^\mu)\, \tilde{\mathcal{P}}_\mu \Big), \tag{2.2}$$

where $x_0 \in M$ is arbitrarily fixed and $x \in M$,[3] is unitary, i.e. $\mathcal{U}^\dagger(x_0, x) := (\mathcal{U}(x, x_0))^\dagger = \mathcal{U}^{-1}(x, x_0) = (\mathcal{U}(x, x_0))^{-1}$ and, via the formulae

$$\tilde{\mathcal{X}} \mapsto \mathcal{X}(x) = \mathcal{U}(x, x_0)(\tilde{\mathcal{X}}) \tag{2.3}$$

$$\tilde{\mathcal{A}}(x) \mapsto \mathcal{A}(x) = \mathcal{U}(x, x_0) \circ (\tilde{\mathcal{A}}(x)) \circ \mathcal{U}^{-1}(x, x_0), \tag{2.4}$$

realizes the transition to the *momentum picture*. Here $\tilde{\mathcal{X}}$ is a state vector in system's Hilbert space of states $\mathcal{F}$ and $\tilde{\mathcal{A}}(x)\colon \mathcal{F} \to \mathcal{F}$ is (observable or not) operator-valued function of $x \in M$ which, in particular, can be polynomial or convergent power series in the field operators

[3] The notation x_0, for a fixed point in M, should not be confused with the zeroth covariant coordinate $\eta_{0\mu} x^\mu$ of x which, following the convention $x_\nu := \eta_{\nu\mu} x^\mu$, is denoted by the same symbol x_0. From the context, it will always be clear whether x_0 refers to a point in M or to the zeroth covariant coordinate of a point $x \in M$.

$\tilde{\varphi}_i(x)$; respectively $\mathcal{X}(x)$ and $\mathcal{A}(x)$ are the corresponding quantities in momentum picture. In particular, the field operators transform as

$$\tilde{\varphi}_i(x) \mapsto \varphi_i(x) = \mathcal{U}(x, x_0) \circ \tilde{\varphi}_i(x) \circ \mathcal{U}^{-1}(x, x_0). \tag{2.5}$$

Notice, in (2.2) the multiplier $(x^\mu - x_0^\mu)$ is regarded as a real parameter (in which $\tilde{\mathcal{P}}_\mu$ is linear). Generally, $\mathcal{X}(x)$ and $\mathcal{A}(x)$ depend also on the point x_0 and, to be quite correct, one should write $\mathcal{X}(x, x_0)$ and $\mathcal{A}(x, x_0)$ for $\mathcal{X}(x)$ and $\mathcal{A}(x)$, respectively. However, in the most situations in the present work, this dependence is not essential or, in fact, is not presented at all. For that reason, we shall *not* indicate it explicitly.

As it was said above, we consider quantum field theories in which the components $\tilde{\mathcal{P}}_\mu$ of the momentum operator commute between themselves and satisfy the Heisenberg relations/equations with the field operators, i.e. we suppose that $\tilde{\mathcal{P}}_\mu$ and $\tilde{\varphi}_i(x)$ satisfy the relations:

$$[\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{P}}_\nu]_{\_} = 0 \tag{2.6}$$

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\mu]_{\_} = \mathrm{i}\hbar \partial_\mu \tilde{\varphi}_i(x). \tag{2.7}$$

Here $[A, B]_\pm := A \circ B \pm B \circ A$, $\circ$ being the composition of mappings sign, is the commutator/anticommutator of operators (or matrices) A and B. The momentum operator $\tilde{\mathcal{P}}_\mu$ commutes with the 'evolution' operator $\mathcal{U}(x, x_0)$ (see below (2.12)) and its inverse,

$$[\tilde{\mathcal{P}}_\mu, \mathcal{U}(x, x_0)]_{\_} = 0 \qquad [\tilde{\mathcal{P}}_\mu, \mathcal{U}^{-1}(x, x_0)]_{\_} = 0, \tag{2.8}$$

due to (2.6) and (2.2). So, the momentum operator remains unchanged in momentum picture, viz. we have (see (2.4) and (2.8))

$$\mathcal{P}_\mu = \tilde{\mathcal{P}}_\mu. \tag{2.9}$$

Since from (2.2) and (2.6) follows

$$\mathrm{i}\hbar \frac{\partial \mathcal{U}(x, x_0)}{\partial x^\mu} = \mathcal{P}_\mu \circ \mathcal{U}(x, x_0) \qquad \mathcal{U}(x_0, x_0) = \mathsf{id}_{\mathcal{F}}, \tag{2.10}$$

we see that, due to (2.3), a state vector $\mathcal{X}(x)$ in momentum picture is a solution of the initial-value problem

$$\mathrm{i}\hbar \frac{\partial \mathcal{X}(x)}{\partial x^\mu} = \mathcal{P}_\mu(\mathcal{X}(x)) \qquad \mathcal{X}(x)|_{x=x_0} = \mathcal{X}(x_0) = \tilde{\mathcal{X}} \tag{2.11}$$

which is a 4-dimensional analogue of a similar problem for the Schrödinger equation in quantum mechanics [20–22].

By virtue of (2.2), or in view of the independence of $\mathcal{P}_\mu$ of x, the solution of (2.11) is

$$\mathcal{X}(x) = \mathcal{U}(x, x_0)(\mathcal{X}(x_0)) = \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu)\mathcal{P}_\mu}(\mathcal{X}(x_0)). \tag{2.12}$$

Thus, if $\mathcal{X}(x_0) = \tilde{\mathcal{X}}$ is an eigenvector of $\mathcal{P}_\mu$ $(= \tilde{\mathcal{P}}_\mu)$ with eigenvalues p_μ,

$$\mathcal{P}_\mu(\mathcal{X}(x_0)) = p_\mu \mathcal{X}(x_0) \quad (= p_\mu \tilde{\mathcal{X}} = \tilde{\mathcal{P}}_\mu(\tilde{\mathcal{X}})), \tag{2.13}$$

we have the following *explicit* form of the state vectors

$$\mathcal{X}(x) = \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu)p_\mu}(\mathcal{X}(x_0)). \tag{2.14}$$

It should clearly be understood, *this is the general form of all state vectors* as they are eigenvectors of all (commuting) observables [3, p. 59], in particular, of the momentum operator.

In momentum picture, all of the field operators happen to be constant in spacetime, i.e.

$$\varphi_i(x) = \mathcal{U}(x, x_0) \circ \tilde{\varphi}_i(x) \circ \mathcal{U}^{-1}(x, x_0) = \varphi_i(x_0) = \tilde{\varphi}_i(x_0) =: \varphi_{(0)\,i}. \tag{2.15}$$

Evidently, a similar result is valid for any (observable or not such) function of the field operators which is polynomial or convergent power series in them and/or their first partial derivatives. However, if $\tilde{\mathcal{A}}(x)$ is an arbitrary operator or depends on the field operators in a different way, then the corresponding to it operator $\mathcal{A}(x)$ according to (2.4) is, generally, not spacetime-constant and depends on the both points x and x_0. As a rules, if $\mathcal{A}(x) = \mathcal{A}(x, x_0)$ is independent of x, we, usually, write $\mathcal{A}$ for $\mathcal{A}(x, x_0)$, omitting all arguments.

It should be noted, the Heisenberg relations (2.7) in momentum picture transform into the identities $\partial_\mu \varphi_i = 0$ meaning that the field operators φ_i in momentum picture are spacetime constant operators (see (2.15)). So, in momentum picture, the Heisenberg relations (2.7) are incorporated in the constancy of the field operators.

Let $\tilde{\mathcal{L}}$ be the system's Lagrangian (in Heisenberg picture). It is supposed to be polynomial or convergent power series in the field operators and their first partial derivatives, i.e. $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\varphi_i(x), \partial_\nu \varphi_i(x))$ with ∂_ν denoting the partial derivative operator relative to the ν^{th} coordinate x^ν. In momentum picture it transforms into

$$\mathcal{L} = \tilde{\mathcal{L}}(\varphi_i(x), y_{j\nu}) \qquad y_{j\nu} = \frac{1}{\mathrm{i}\hbar}[\varphi_j, \mathcal{P}_\nu]_{\_}, \tag{2.16}$$

i.e. in momentum picture one has simply to replace the field operators in Heisenberg picture with their values at a fixed point x_0 and the partial derivatives $\partial_\nu \tilde{\varphi}_j(x)$ in Heisenberg picture with the above-defined quantities $y_{j\nu}$. The (constant) field operators φ_i satisfy the following *algebraic Euler-Lagrange equations in momentum picture*:[4]

$$\Big\{ \frac{\partial \tilde{\mathcal{L}}(\varphi_j, y_{l\nu})}{\partial \varphi_i} - \frac{1}{\mathrm{i}\hbar} \Big[\frac{\partial \tilde{\mathcal{L}}(\varphi_j, y_{l\nu})}{y_{i\mu}}, \mathcal{P}_\mu \Big]_{\_} \Big\} \Big|_{y_{j\nu} = \frac{1}{\mathrm{i}\hbar}[\varphi_j, \mathcal{P}_\nu]_{\_}} = 0. \tag{2.17}$$

Since $\mathcal{L}$ is supposed to be polynomial or convergent power series in its arguments, the equations (2.17) are *algebraic*, not differential, ones. This result is a natural one in view of (2.15).

Suppose a quantum system under consideration possesses a charge (e.g. electric one) and angular momentum, described by respectively the current operator $\tilde{\mathcal{J}}_\mu(x)$ and (total) angular momentum tensorial density operator

$$\tilde{\mathcal{M}}^\lambda_{\mu\nu}(x) = -\tilde{\mathcal{M}}^\lambda_{\nu\mu}(x) = x_\mu \tilde{\mathcal{T}}^\lambda{}_\nu - x_\nu \tilde{\mathcal{T}}^\lambda{}_\mu + \tilde{\mathcal{S}}^\lambda_{\mu\nu}(x) \tag{2.18}$$

with $x_\nu := \eta_{\nu\mu} x^\mu$ and $\tilde{\mathcal{S}}^\lambda_{\mu\nu}(x) = -\tilde{\mathcal{S}}^\lambda_{\nu\mu}(x)$ being the spin angular momentum (density) operator. The (constant, time-independent) conserved quantities corresponding to them, the charge operator $\tilde{\mathcal{Q}}$ and total angular momentum operator $\tilde{\mathcal{M}}_{\mu\nu}$, respectively are

$$\tilde{\mathcal{Q}} := \frac{1}{c} \int\limits_{x^0=\text{const}} \tilde{\mathcal{J}}_0(x)\, \mathrm{d}^3\boldsymbol{x} \tag{2.19}$$

$$\tilde{\mathcal{M}}_{\mu\nu} = \tilde{\mathcal{L}}_{\mu\nu}(x) + \tilde{\mathcal{S}}_{\mu\nu}(x), \tag{2.20}$$

[4] In (2.17) and similar expressions appearing further, the derivatives of functions of operators with respect to operator arguments are calculated in the same way as if the operators were ordinary (classical) fields/functions, only the order of the arguments should not be changed. This is a silently accepted practice in the literature [2, 3]. In the most cases such a procedure is harmless, but it leads to the problem of non-unique definitions of the quantum analogues of the classical conserved quantities, like the energy-momentum and charge operators. For some details on this range of problems in quantum field theory, see [17]; in particular, the *loc. cit.* contains an example of a Lagrangian whose field equations are *not* the Euler-Lagrange equations (2.17) obtained as just described.

where

$$\tilde{\mathcal{L}}_{\mu\nu}(x) := \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \{x_\mu \tilde{\mathcal{T}}^0{}_\nu(x) - x_\nu \tilde{\mathcal{T}}^0{}_\mu(x)\} \,\mathrm{d}^3\boldsymbol{x} \tag{2.21a}$$

$$\tilde{\mathcal{S}}_{\mu\nu}(x) := \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \tilde{\mathcal{S}}^0_{\mu\nu}(x) \,\mathrm{d}^3\boldsymbol{x} \tag{2.21b}$$

are the orbital and spin, respectively, angular momentum operators (in Heisenberg picture). Notice, we write $\tilde{\mathcal{L}}_{\mu\nu}(x)$ and $\tilde{\mathcal{S}}_{\mu\nu}(x)$, but, as a result of (2.21), these operators may depend only on the zeroth (time) coordinate of $x \in M$. When working in momentum picture, in view of (2.4), the following representations turn to be useful:

$$\mathcal{P}_\mu = \tilde{\mathcal{P}}_\mu = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \mathcal{T}_{0\mu} \circ \mathcal{U}(x, x_0) \,\mathrm{d}^3\boldsymbol{x} \tag{2.22}$$

$$\tilde{\mathcal{Q}} = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \mathcal{J}_0 \circ \mathcal{U}(x, x_0) \,\mathrm{d}^3\boldsymbol{x} \tag{2.23}$$

$$\tilde{\mathcal{L}}_{\mu\nu}(x) = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \{x_\mu \mathcal{T}^0{}_\nu - x_\nu \mathcal{T}^0{}_\mu\} \circ \mathcal{U}(x, x_0) \,\mathrm{d}^3\boldsymbol{x} \tag{2.24}$$

$$\tilde{\mathcal{S}}_{\mu\nu}(x) = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \mathcal{S}^0_{\mu\nu} \circ \mathcal{U}(x, x_0) \,\mathrm{d}^3\boldsymbol{x}. \tag{2.25}$$

These expressions will be employed essentially in the present paper.

The conservation laws $\frac{\mathrm{d}\,\tilde{\mathcal{Q}}}{\mathrm{d}x^0} = 0$ and $\frac{\mathrm{d}\,\tilde{\mathcal{M}}_{\mu\nu}}{\mathrm{d}x^0} = 0$ (or, equivalently, $\partial^\mu \tilde{\mathcal{J}}_\mu = 0$ and $\partial_\lambda \tilde{\mathcal{M}}^\lambda_{\mu\nu} = 0$), can be rewritten as

$$\partial_\mu \tilde{\mathcal{Q}} = 0 \qquad \partial_\lambda \tilde{\mathcal{M}}_{\mu\nu} = 0 \tag{2.26}$$

since (2.19)–(2.21) imply $\partial_a \tilde{\mathcal{Q}} = 0$ and $\partial_a \tilde{\mathcal{M}}_{\mu\nu} = 0$ for $a = 1, 2, 3$.

As a result of the skewsymmetry of the operators (2.20) and (2.21) in the subscripts μ and ν, their spacial components form a (pseudo-)vectorial operators. If e^{abc}, $a, b, c = 1, 2, 3$, denotes the 3-dimensional Levi-Civita (totally) antisymmetric symbol, we put $\tilde{\boldsymbol{M}} := (\tilde{\boldsymbol{M}}^1, \tilde{\boldsymbol{M}}^2, \tilde{\boldsymbol{M}}^3)$ with $\tilde{\boldsymbol{M}}^a := e^{abc} \tilde{\mathcal{M}}_{bc}$ and similarly for the orbital and spin angular momentum operators. Then (2.20) and the below written equation (2.37) imply

$$\tilde{\boldsymbol{M}} = \tilde{\boldsymbol{L}}(x) + \tilde{\boldsymbol{S}}(x) \tag{2.27}$$

$$\boldsymbol{M}(x, x_0) = \tilde{\boldsymbol{L}}(x) + (\boldsymbol{x} - \boldsymbol{x}_0) \times \boldsymbol{P} + \tilde{\boldsymbol{S}}(x), \tag{2.28}$$

where $\boldsymbol{x} := (x^1, x^2, x^3) = -(x_1, x_2, x_3)$, $\times$ denotes the Euclidean cross product, and $\boldsymbol{P} := (\mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3) = -(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$. Obviously, the correction in (2.28) to $\tilde{\boldsymbol{M}}$ can be interpreted as a one due to an additional orbital angular momentum when the origin, with respect to which it is determined, is change from x to x_0.

The consideration of $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{M}}_{\mu\nu}$ as generators of constant phase transformations and 4-rotations, respectively, leads to the following relations [1, 2, 16]

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{Q}}]_{\_} = \varepsilon(\tilde{\varphi}_i) q_i \tilde{\varphi}_i(x) \tag{2.29}$$

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{M}}_{\mu\nu}]_{\_} = \mathrm{i}\hbar\{x_\mu \partial_\nu \tilde{\varphi}_i(x) - x_\nu \partial_\mu \tilde{\varphi}_i(x) + I^j_{i\mu\nu} \tilde{\varphi}_j(x)\}. \tag{2.30}$$

Here: $q_i = \mathrm{const}$ is the charge of the i^{th} field, $q_j = q_i$ if $\tilde{\varphi}_j = \tilde{\varphi}^\dagger_i$, $\varepsilon(\tilde{\varphi}_i) = 0$ if $\tilde{\varphi}^\dagger_i = \tilde{\varphi}_i$, $\varepsilon(\tilde{\varphi}_i) = \pm 1$ if $\tilde{\varphi}^\dagger_i \neq \tilde{\varphi}_i$ with $\varepsilon(\tilde{\varphi}_i) + \varepsilon(\tilde{\varphi}^\dagger_i) = 0$, and the constants $I^j_{i\mu\nu} = -I^j_{i\nu\mu}$ characterize

the transformation properties of the field operators under 4-rotations. (If $\varepsilon(\tilde{\varphi}_i) \neq 0$, it is a convention whether to put $\varepsilon(\tilde{\varphi}_i) = +1$ or $\varepsilon(\tilde{\varphi}_i) = -1$ for a fixed i.) Besides, the operators (2.19)–(2.21) are Hermitian,

$$\tilde{\mathcal{Q}}^\dagger = \tilde{\mathcal{Q}}, \quad \tilde{\mathcal{M}}^\dagger_{\mu\nu} = \tilde{\mathcal{M}}_{\mu\nu}, \quad \tilde{\mathcal{L}}^\dagger_{\mu\nu} = \tilde{\mathcal{L}}_{\mu\nu}, \quad \tilde{\mathcal{S}}^\dagger_{\mu\nu} = \tilde{\mathcal{S}}_{\mu\nu}, \tag{2.31}$$

and satisfy the relations[5]

$$[\tilde{\mathcal{Q}}, \tilde{\mathcal{P}}_\mu]_{\_} = 0 \tag{2.32}$$

$$[\tilde{\mathcal{M}}_{\mu\nu}, \tilde{\mathcal{P}}_\lambda]_{\_} = -\mathrm{i}\hbar\{\eta_{\lambda\mu}\tilde{\mathcal{P}}_\nu - \eta_{\lambda\nu}\tilde{\mathcal{P}}_\mu\}. \tag{2.33}$$

Combining the last two equalities with (2.2) and (2.6), we, after a simple algebraic calculations, obtain[6]

$$[\tilde{\mathcal{Q}}, \mathcal{U}(x,x_0)]_{\_} = 0 \tag{2.34}$$

$$[\tilde{\mathcal{M}}_{\mu\nu}, \mathcal{U}(x,x_0)]_{\_} = -\{(x_\mu - x_{0\,\mu})\tilde{\mathcal{P}}_\nu - (x_\nu - x_{0\,\nu})\tilde{\mathcal{P}}_\mu\} \circ \mathcal{U}(x,x_0). \tag{2.35}$$

Consequently, in accord with (2.4), in momentum picture the charge and angular momentum operators respectively are

$$\mathcal{Q}(x) = \tilde{\mathcal{Q}} =: \mathcal{Q} \tag{2.36}$$

$$\begin{aligned}\mathcal{M}_{\mu\nu} &= \mathcal{U}(x,x_0)\circ\tilde{\mathcal{M}}_{\mu\nu}\circ\mathcal{U}^{-1}(x,x_0) = \tilde{\mathcal{M}}_{\mu\nu} + [\mathcal{U}(x,x_0), \tilde{\mathcal{M}}_{\mu\nu}]_{\_}\circ\mathcal{U}^{-1}(x,x_0)\\ &= \tilde{\mathcal{M}}_{\mu\nu} + (x_\mu - x_{0\,\mu})\mathcal{P}_\nu - (x_\nu - x_{0\,\nu})\mathcal{P}_\mu\\ &= \tilde{\mathcal{L}}_{\mu\nu} + (x_\mu - x_{0\,\mu})\mathcal{P}_\nu - (x_\nu - x_{0\,\nu})\mathcal{P}_\mu + \tilde{\mathcal{S}}_{\mu\nu} = \mathcal{L}_{\mu\nu} + \mathcal{S}_{\mu\nu},\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}\tilde{\mathcal{L}}_{\mu\nu}(x) &:= \mathcal{U}(x,x_0)\circ\mathcal{L}_{\mu\nu}(x)\circ\mathcal{U}^{-1}(x,x_0)\\ \tilde{\mathcal{S}}_{\mu\nu}(x) &:= \mathcal{U}(x,x_0)\circ\mathcal{S}_{\mu\nu}(x)\circ\mathcal{U}^{-1}(x,x_0)\end{aligned} \tag{2.38}$$

and (2.9) was taken into account. Notice, the correction to $\tilde{\mathcal{M}}_{\mu\nu}$ on the r.h.s. of (2.37) is typical for the one of classical orbital angular momentum when the origin, with respect to which it is determined, is changed from x to x_0.[7]

---

[5] The author is completely aware of the fact that in the literature, for instance in [3, p. 77, eq. (2-87)] or in [4, eq. (2.187)], the relation (2.33) is written with an opposite sign, i.e. with $+\mathrm{i}\hbar$ instead of $-\mathrm{i}\hbar$ on its r.h.s. (In this case (2.33) is part of the commutation relations characterizing the Lie algebra of the Poincaré group — see, e.g., [8, pp. 143–147] or [9, sect. 7.1].) However, such a choice of the sign in (2.33) contradicts to the explicit form of $\tilde{\mathcal{P}}_\mu$ and $\tilde{\mathcal{L}}_{\mu\nu}$ in terms of creation and annihilation operators (see sections 6 and 7) in the framework of Lagrangian formalism. For this reason and since the relation (2.33) is external to the Lagrangian formalism, we accept (2.33) as it is written below. In connection with (2.33) — see below equation (7.8), (7.9) and (7.13).

[6] To derive equation (2.35), notice that (2.33) implies $[\tilde{\mathcal{M}}_{\mu\nu}, \tilde{\mathcal{P}}_{\mu_1}\circ\cdots\circ\tilde{\mathcal{P}}_{\mu_n}]_{\_} = -\sum_{i=1}^{n}(\eta_{\mu\mu_i}\tilde{\mathcal{P}}_\nu - \eta_{\nu\mu_i}\tilde{\mathcal{P}}_\mu)\tilde{\mathcal{P}}_{\mu_1}\circ\cdots\circ\tilde{\mathcal{P}}_{\mu_{i-1}}\circ\tilde{\mathcal{P}}_{\mu_{i+1}}\circ\cdots\circ\tilde{\mathcal{P}}_{\mu_n}$, due to $[A, B\circ C]_{\_} = [A,B]_{\_}\circ C + B\circ[A,C]_{\_}$, and expand the exponent in (2.2) into a power series. More generally, if $[A(x), \tilde{\mathcal{P}}_\mu]_{\_} = B_\mu(x)$ with $[B_\mu(x), \tilde{\mathcal{P}}_\nu]_{\_} = 0$, then $[A(x), \mathcal{U}(x,x_0)]_{\_} = \frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu)B_\mu(x)\circ\mathcal{U}(x,x_0)$; in particular, $[A(x), \tilde{\mathcal{P}}_\mu]_{\_} = 0$ implies $[A(x), \mathcal{U}(x,x_0)]_{\_} = 0$. Notice, we consider $(x^\mu - x_0^\mu)$ as a real parameter by which the corresponding operators are multiplied and which operators are supposed to be linear in it.

[7] In Section 7, it will be proved that, for massive free vector fields (and for massless free vector fields under some conditions), holds the equation

$$[\tilde{\mathcal{S}}_{\mu\nu}, \mathcal{P}_\lambda]_{\_} = 0, \tag{2.39}$$

In momentum picture, by virtue of (2.4), the relations (2.29) and (2.30) respectively read ($\varepsilon(\varphi) := \varepsilon(\tilde{\varphi})$)

$$[\varphi_i, \mathcal{Q}]_{\_} = \varepsilon(\varphi_i) q \varphi_i \tag{2.43}$$

$$[\varphi_i, \mathcal{M}_{\mu\nu}(x, x_0)]_{\_} = x_\mu [\varphi_i, \mathcal{P}_\nu]_{\_} - x_\nu [\varphi_i, \mathcal{P}_\mu]_{\_} + \mathrm{i}\hbar I^j_{i\mu\nu} \varphi_j. \tag{2.44}$$

The first of these equation is evident. To derive the second one, we notice that, by virtue of the Heisenberg relations/equations (2.7), the equality (2.30) is equivalent to

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{M}}_{\mu\nu}]_{\_} = x_\mu [\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\nu]_{\_} - x_\nu [\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\mu]_{\_} + \mathrm{i}\hbar I^j_{i\mu\nu} \tilde{\varphi}_j(x) \tag{2.45}$$

from where (2.44) follows.

It should be emphasized, the Heisenberg relations (2.29) and (2.30), as well as the commutation relations (2.32) and (2.33), are external to the Lagrangian formalism. For this reason, one should be quite careful when applying them unless they are explicitly proved in the framework of Lagrangian scheme.

3. Description of free vector field in momentum picture

A vector field $\mathcal{U}$ is described by four operators $\tilde{\mathcal{U}}_\mu := \tilde{\mathcal{U}}_\mu(x)$, called its components, which transform as components of a 4-vector under Poincaré transformations. The operators $\tilde{\mathcal{U}}_\mu$ are Hermitian, $\tilde{\mathcal{U}}^\dagger_\mu = \tilde{\mathcal{U}}_\mu$, for a neutral field and non-Hermitian, $\tilde{\mathcal{U}}^\dagger_\mu \neq \tilde{\mathcal{U}}_\mu$, for a charged one. Since the consideration of $\tilde{\mathcal{U}}_0, \dots, \tilde{\mathcal{U}}_3$ as independent scalar fields meets as an obstacle the non-positivity of the energy (see, e.g., [1, § 4.1] or [23, § 2a]), the Lagrangian of a free vector field is represented as a sum of the Lagrangians, corresponding to $\tilde{\mathcal{U}}_0, \dots, \tilde{\mathcal{U}}_3$ considered as independent scalar fields, and a 'correction' term(s) ensuring the energy positivity (and, in fact, defining $\tilde{\mathcal{U}}$ as spin 1 quantum field). As pointed in the discussion in [1, § 4.1 and § 5.3], the Lagrangian of a free vector field (and, possibly, conditions additional to the Lagrangian formalism) can be chosen in different ways, which lead to identical theories, i.e. to coinciding field equations and dynamical variables.[8]

Between a number of possibilities for describing a *massive* vector field of mass $m \neq 0$, we choose the Lagrangian as [15]

$$\tilde{\mathcal{L}} = \frac{m^2c^4}{1+\tau(\tilde{\mathcal{U}})} \tilde{\mathcal{U}}^\dagger_\mu \circ \tilde{\mathcal{U}}^\mu - \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} (\partial_\mu \tilde{\mathcal{U}}^\dagger_\nu) \circ (\partial^\mu \tilde{\mathcal{U}}^\nu) + \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} (\partial_\mu \tilde{\mathcal{U}}^{\mu\dagger}) \circ (\partial_\nu \tilde{\mathcal{U}}^\nu), \tag{3.1}$$

where the function τ takes care of is the field neutral (Hermitian) or charged (non-Hermitian) according to

$$\tau(\tilde{\mathcal{U}}) := \begin{cases} 1 & \text{for } \tilde{\mathcal{U}}^\dagger_\mu = \tilde{\mathcal{U}}_\mu \text{ (Hermitian (neutral) field)} \\ 0 & \text{for } \tilde{\mathcal{U}}^\dagger_\mu \neq \tilde{\mathcal{U}}_\mu \text{ (non-Hermitian (charged) field)} \end{cases}. \tag{3.2}$$

which implies

$$[\tilde{\mathcal{S}}_{\mu\nu}, \mathcal{U}(x, s_0)]_{\_} = 0. \tag{2.40}$$

Amongst other things, from here follow the equations

$$\mathcal{S}_{\mu\nu} = \tilde{\mathcal{S}}_{\mu\nu} \tag{2.41}$$

$$\mathcal{L}_{\mu\nu} = \tilde{\mathcal{L}}_{\mu\nu} + (x_\mu - x_{0\,\mu}) \mathcal{P}_\nu - (x_\nu - x_{0\,\nu}) \mathcal{P}_\mu. \tag{2.42}$$

[8] This may not be the case when *interacting* fields are considered.

Since

$$(\partial_\mu \tilde{\mathcal{U}}^{\mu\dagger}) \circ (\partial_\nu \tilde{\mathcal{U}}^\nu) - (\partial_\mu \tilde{\mathcal{U}}_\nu^\dagger) \circ (\partial^\nu \tilde{\mathcal{U}}^\mu) = \partial_\mu \{ \tilde{\mathcal{U}}^{\mu\dagger} \circ (\partial_\nu \tilde{\mathcal{U}}^\nu) - \tilde{\mathcal{U}}_\nu^\dagger \circ (\partial^\nu \tilde{\mathcal{U}}^\mu) \},$$

the theory arising from the Lagrangian (3.1) is equivalent to the one build from [4, 16]

$$\begin{aligned} \tilde{\mathcal{L}} &= \frac{m^2c^4}{1+\tau(\tilde{\mathcal{U}})} \tilde{\mathcal{U}}_\mu^\dagger \circ \tilde{\mathcal{U}}^\mu - \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} (\partial_\mu \tilde{\mathcal{U}}_\nu^\dagger) \circ (\partial^\mu \tilde{\mathcal{U}}^\nu) + \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} (\partial_\mu \tilde{\mathcal{U}}_\nu^\dagger) \circ (\partial^\nu \tilde{\mathcal{U}}^\mu) \\ &= \frac{m^2c^4}{1+\tau(\tilde{\mathcal{U}})} \tilde{\mathcal{U}}_\mu^\dagger \circ \tilde{\mathcal{U}}^\mu - \frac{c^2\hbar^2}{2(1+\tau(\tilde{\mathcal{U}}))} \tilde{\mathcal{F}}_{\mu\nu}^\dagger \circ \tilde{\mathcal{F}}^{\mu\nu}, \end{aligned} \tag{3.3}$$

where

$$\tilde{\mathcal{F}}_{\mu\nu} := \partial_\mu \tilde{\mathcal{U}}_\nu - \partial_\nu \tilde{\mathcal{U}}_\mu. \tag{3.4}$$

The first two terms in (3.1) (or in the first row in (3.3)) correspond to a sum of four independent Lagrangians for the components $\tilde{\mathcal{U}}_0, \dots, \tilde{\mathcal{U}}_3$, considered as free scalar fields [1, 13, 15]. The remaining terms in (3.1) or (3.3) represent the afore-mentioned 'correction' which reduces the independent components (degrees of freedom) of a vector field from 4 to 3 and ensures the positivity of the field's energy [1, 15, 16].

Before proceeding with the description in momentum picture, we notice that the Euler-Lagrange equations for the Lagrangians (3.1) and (3.3) coincide and are

$$0 = m^2c^2 \tilde{\mathcal{U}}_\mu + \hbar^2 \tilde{\Box}(\tilde{\mathcal{U}}_\mu) - \hbar^2 \partial_\mu (\partial^\lambda \tilde{\mathcal{U}}_\lambda) = m^2c^2 \tilde{\mathcal{U}}_\mu + \hbar^2 \partial^\lambda \tilde{\mathcal{F}}_{\lambda\mu} \tag{3.5a}$$

$$0 = m^2c^2 \tilde{\mathcal{U}}_\mu^\dagger + \hbar^2 \tilde{\Box}(\tilde{\mathcal{U}}_\mu^\dagger) - \hbar^2 \partial_\mu (\partial^\lambda \tilde{\mathcal{U}}_\lambda^\dagger) = m^2c^2 \tilde{\mathcal{U}}_\mu^\dagger + \hbar^2 \partial^\lambda \tilde{\mathcal{F}}_{\lambda\mu}^\dagger, \tag{3.5b}$$

where $\tilde{\Box} := \partial_\lambda \partial^\lambda$ is the D'Alembert operator (in Heisenberg picture). For $m \neq 0$, these equations, known as the *Proca equations* for $\tilde{\mathcal{U}}_\mu$ and $\tilde{\mathcal{U}}_\mu^\dagger$, can be written equivalently as

$$(m^2c^2 + \hbar^2 \tilde{\Box}) \tilde{\mathcal{U}}_\mu = 0 \qquad (m^2c^2 + \hbar^2 \tilde{\Box}) \tilde{\mathcal{U}}_\mu^\dagger = 0 \tag{3.6a}$$

$$\partial^\mu \tilde{\mathcal{U}}_\mu = 0 \qquad \partial^\mu \tilde{\mathcal{U}}_\mu^\dagger = 0, \tag{3.6b}$$

due to $\partial^\mu \partial^\nu \tilde{\mathcal{F}}_{\mu\nu} \equiv 0$ and $m \neq 0$, and show that there is a bijective correspondence between $\tilde{\mathcal{U}}_\mu$ and $\tilde{\mathcal{F}}_{\mu\nu}$ (in the case $m \neq 0$) [23, § 2]. Therefore the field operators $\tilde{\mathcal{U}}_\mu$ and $\tilde{\mathcal{U}}_\mu^\dagger$ are solutions of the *Klein-Gordon equations* (3.6a) with mass m ($\neq 0$) and satisfy the conditions (3.6b), known as the *Lorenz conditions*. We shall say that a vector field satisfies the Lorenz condition, if the equations (3.6b) hold for it.

According to (2.16), the Lagrangians (3.1) and (3.3) in momentum picture are

$$\mathcal{L} = \frac{m^2c^4}{1+\tau(\mathcal{U})} \mathcal{U}_\mu^\dagger \circ \mathcal{U}^\mu + \frac{c^2}{1+\tau(\mathcal{U})} \{ [\mathcal{U}_\nu^\dagger, \mathcal{P}_\mu]_\_ \circ [\mathcal{U}^\nu, \mathcal{P}^\mu]_\_ - [\mathcal{U}^{\mu\dagger}, \mathcal{P}_\mu]_\_ \circ [\mathcal{U}^\nu, \mathcal{P}_\nu]_\_ \} \tag{3.7}$$

$$\begin{aligned} \mathcal{L} &= \frac{m^2c^4}{1+\tau(\mathcal{U})} \mathcal{U}_\mu^\dagger \circ \mathcal{U}^\mu + \frac{c^2}{1+\tau(\mathcal{U})} \{ [\mathcal{U}_\nu^\dagger, \mathcal{P}_\mu]_\_ \circ [\mathcal{U}^\nu, \mathcal{P}^\mu]_\_ - [\mathcal{U}_\nu^\dagger, \mathcal{P}_\mu]_\_ \circ [\mathcal{U}^\mu, \mathcal{P}^\nu]_\_ \} \\ &= \frac{m^2c^4}{1+\tau(\mathcal{U})} \mathcal{U}_\mu^\dagger \circ \mathcal{U}^\mu - \frac{c^2\hbar^2}{2(1+\tau(\mathcal{U}))} \mathcal{F}_{\mu\nu}^\dagger \circ \mathcal{F}^{\mu\nu}, \end{aligned} \tag{3.8}$$

respectively, where

$$\mathcal{U}_\mu(x) := \mathcal{U}(x, x_0) \circ \tilde{\mathcal{U}}_\mu(x) \circ \mathcal{U}^{-1}(x, x_0) \quad \mathcal{U}_\mu^\dagger(x) := \mathcal{U}(x, x_0) \circ \tilde{\mathcal{U}}_\mu^\dagger(x) \circ \mathcal{U}^{-1}(x, x_0) \tag{3.9}$$

$$\tau(\mathcal{U}) := \begin{cases} 1 & \text{for } \mathcal{U}_\mu^\dagger = \mathcal{U}_\mu \text{ (Hermitian (neutral) field)} \\ 0 & \text{for } \mathcal{U}_\mu^\dagger \neq \mathcal{U}_\mu \text{ (non-Hermitian (charged) field)} \end{cases} = \tau(\tilde{\mathcal{U}}) \tag{3.10}$$

$$\mathcal{F}_{\mu\nu}(x) = \mathcal{U}(x, x_0) \circ (\tilde{\mathcal{F}}_{\mu\nu}(x)) \circ \mathcal{U}^{-1}(x, x_0) = -\frac{1}{i\hbar} \{ [\mathcal{U}_\mu, \mathcal{P}_\nu]_\_ - [\mathcal{U}_\nu, \mathcal{P}_\mu]_\_ \}. \tag{3.11}$$

Regarding $\mathcal{U}_\mu$ and $\mathcal{U}_\mu^\dagger$ as independent variables, from (3.7), we get[9]

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\mathcal{U}^\mu} &= \frac{\partial\tilde{\mathcal{L}}}{\partial\mathcal{U}^\mu} = m^2c^4\mathcal{U}_\mu^\dagger \qquad \pi_{\mu\lambda} := \frac{\partial\mathcal{L}}{\partial y^{\mu\lambda}} = \mathrm{i}\hbar c^2[\mathcal{U}_\mu^\dagger, \mathcal{P}_\lambda]_{\_} - \mathrm{i}\hbar c^2\eta_{\mu\lambda}[\mathcal{U}^{\nu\dagger}, \mathcal{P}_\nu]_{\_} \\
\frac{\partial\mathcal{L}}{\partial\mathcal{U}^{\mu\dagger}} &= \frac{\partial\tilde{\mathcal{L}}}{\partial\mathcal{U}^{\mu\dagger}} = m^2c^4\mathcal{U}_\mu \qquad \pi_{\mu\lambda}^\dagger := \frac{\partial\mathcal{L}}{\partial y^{\mu\lambda\dagger}} = \mathrm{i}\hbar c^2[\mathcal{U}_\mu, \mathcal{P}_\lambda]_{\_} - \mathrm{i}\hbar c^2\eta_{\mu\lambda}[\mathcal{U}^\nu, \mathcal{P}_\nu]_{\_}
\end{aligned}
\tag{3.12}
$$

with

$$
y_{\mu\lambda} := \frac{1}{\mathrm{i}\hbar}[\mathcal{U}_\mu, \mathcal{P}_\lambda]_{\_} \quad y_{\mu\lambda}^\dagger := \frac{1}{\mathrm{i}\hbar}[\mathcal{U}_\mu^\dagger, \mathcal{P}_\lambda]_{\_}.
$$

Therefore the field equations (2.17) now read

$$
m^2c^2\mathcal{U}_\mu - [[\mathcal{U}_\mu, \mathcal{P}_\lambda]_{\_}, \mathcal{P}^\lambda]_{\_} + [[\mathcal{U}_\nu, \mathcal{P}^\nu]_{\_}, \mathcal{P}^\mu]_{\_} = 0 \tag{3.13a}
$$

$$
m^2c^2\mathcal{U}_\mu^\dagger - [[\mathcal{U}_\mu^\dagger, \mathcal{P}_\lambda]_{\_}, \mathcal{P}^\lambda]_{\_} + [[\mathcal{U}_\nu^\dagger, \mathcal{P}^\nu]_{\_}, \mathcal{P}^\mu]_{\_} = 0 \tag{3.13b}
$$

or, using the notation (3.4),

$$
m^2c^2\mathcal{U}_\mu - \mathrm{i}\hbar[\mathcal{F}_{\lambda\mu}, \mathcal{P}^\lambda]_{\_} = 0 \quad m^2c^2\mathcal{U}_\mu^\dagger - \mathrm{i}\hbar[\mathcal{F}_{\lambda\mu}^\dagger, \mathcal{P}^\lambda]_{\_} = 0. \tag{3.14}
$$

These are the systems of the *Proca equations in momentum picture* for a massive free spin 1 (vector) fields.

Since the equality $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$ (valid when applied on C^2 functions or operators) in momentum picture takes the form (see (2.4))

$$
[[\cdot, \mathcal{P}_\mu]_{\_}, \mathcal{P}_\nu]_{\_} = [[\cdot, \mathcal{P}_\nu]_{\_}, \mathcal{P}_\mu]_{\_}, \tag{3.15}
$$

from (3.14), (3.15) and $\mathcal{F}_{\mu\nu} = -\mathcal{F}_{\nu\mu}$ follow the equalities

$$
m^2[\mathcal{U}_\mu, \mathcal{P}^\mu]_{\_} = 0 \quad m^2[\mathcal{U}_\mu^\dagger, \mathcal{P}^\mu]_{\_} = 0. \tag{3.16}
$$

Consequently, in the massive case, i.e. $m \neq 0$, the system of Proca equations (3.13) splits into the system of Klein-Gordon equations in momentum picture [13]

$$
m^2c^2\mathcal{U}_\mu - [[\mathcal{U}_\mu, \mathcal{P}_\nu]_{\_}, \mathcal{P}^\nu]_{\_} = 0 \quad m^2c^2\mathcal{U}_\mu^\dagger - [[\mathcal{U}_\mu^\dagger, \mathcal{P}_\nu]_{\_}, \mathcal{P}^\nu]_{\_} = 0 \tag{3.17}
$$

and the system of *Lorenz conditions*

$$
[\mathcal{U}_\mu, \mathcal{P}^\mu]_{\_} = 0 \quad [\mathcal{U}_\mu^\dagger, \mathcal{P}^\mu]_{\_} = 0 \tag{3.18}
$$

for the field operators $\mathcal{U}_\mu$ and $\mathcal{U}_\mu^\dagger$. This result is a momentum picture analogue of (3.6). From technical point of view, it is quite important as it allows a partial application of most of the results obtained for free scalar fields, satisfying (systems of) Klein-Gordon equation(s), to the case of massive vector fields. Evidently, for the solutions of (3.17)–(3.18), the Lagrangian (3.7) reduces to[10]

$$
\mathcal{L} = \frac{m^2c^4}{1+\tau(\mathcal{U})}\mathcal{U}_\mu^\dagger \circ \mathcal{U}^\mu + \frac{c^2}{1+\tau(\mathcal{U})}[\mathcal{U}_\nu^\dagger, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^\nu, \mathcal{P}^\mu]_{\_}, \tag{3.19}
$$

which equals to a sum of four Lagrangians corresponding to $\mathcal{U}_0, \dots, \mathcal{U}_3$ considered as free scalar fields.

---

[9] The derivatives in (3.12) are calculating according to the classical rules of commuting variables, which requires additional rules for ordering the operators in the expressions for dynamical variables; for details, see [17]. The Lagrangian (3.8) has different derivatives, but leads to the same field equations and dynamical variables and, for this reason, will not be considered further in this work.

[10] The same result holds, up to a full divergence, for the Lagrangian (3.8) too.

The above consideration show that the Lagrangian theory of massive free vector field can be constructed equivalently from the Lagrangian (3.19) under the additional conditions (3.18). This procedure is realized in Heisenberg picture in [1].

Consider now the above theory in the massless case, i.e. for $m = 0$. It is easily seen, all of the above conclusions remain valid in the massless case too with one very important exception. Namely, in it the equations (3.16) are identically valid and, consequently, in this case the massless Proca equations, i.e. (3.13) with $m = 0$, do *not* imply the Klein-Gordon equations (3.17) and the Lorenz conditions (3.18).[11] However, one can verify, e.g. in momentum representation in Heisenberg picture, that the Lorenz conditions (3.18) are compatible with the massless Proca equations (3.13) with $m = 0$; said differently, the system of Klein-Gordon equations (3.17) with $m = 0$ and Lorenz conditions (3.18), i.e.

$$[[\mathcal{U}_\mu, \mathcal{P}_\nu]_\_, \mathcal{P}^\nu]_\_ = 0 \quad [[\mathcal{U}^\dagger_\mu, \mathcal{P}_\nu]_\_, \mathcal{P}^\nu]_\_ = 0 \tag{3.20}$$

$$[\mathcal{U}_\mu, \mathcal{P}^\mu]_\_ = 0 \qquad [\mathcal{U}^\dagger_\mu, \mathcal{P}^\mu]_\_ = 0, \tag{3.21}$$

does not contain contradictions and possesses non-trivial solutions. Moreover, one can consider this system of equations as the one describing a free electromagnetic field in Lorenz gauge *before* second quantization, i.e. before imposing a suitable commutation relations between the field's components.

For these reasons, in the present investigation, with an exception of Sect. 13, we shall consider a *quantum field theory build according to the Lagrangian formalism arising from the Lagrangian* (3.7) *to which, in the massless case, are added the Lorenz conditions* (3.18) as additional requirements. In other words, with an exception of Sect. 13, vector fields satisfying the Lorenz conditions will be explored in this work.

A free vector field possesses energy-momentum, (possibly vanishing) charge, and angular momentum. The corresponding to them density operators, the energy-momentum tensor $\tilde{\mathcal{T}}_{\mu\nu}$, current density $\tilde{\mathcal{J}}_\mu$ and (total) angular momentum density $\tilde{\mathcal{M}}^\lambda_{\mu\nu}$, in Heisenberg picture for the Lagrangian (3.7) are as follows:[12]

$$\begin{aligned}
\tilde{\mathcal{T}}_{\mu\nu} &= \frac{1}{1+\tau(\tilde{\mathcal{U}})}\{\,\tilde{\pi}_{\lambda\mu}\circ(\partial_\nu\tilde{\mathcal{U}}^\lambda) + (\partial_\nu\tilde{\mathcal{U}}^{\lambda\dagger})\circ\tilde{\pi}^\dagger_{\lambda\mu}\} - \eta_{\mu\nu}\,\tilde{\mathcal{L}} \\
&= -\frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})}\{(\partial_\mu\tilde{\mathcal{U}}^\dagger_\lambda)\circ(\partial_\nu\tilde{\mathcal{U}}^\lambda) + (\partial_\nu\tilde{\mathcal{U}}^\dagger_\lambda)\circ(\partial_\mu\tilde{\mathcal{U}}^\lambda)\} - \eta_{\mu\nu}\,\tilde{\mathcal{L}} = \tilde{\mathcal{T}}_{\nu\mu}
\end{aligned} \tag{3.22}$$

$$\tilde{\mathcal{J}}_\mu = \frac{q}{\mathrm{i}\hbar}\{\,\tilde{\pi}_{\lambda\mu}\circ\tilde{\mathcal{U}}^\lambda - \tilde{\mathcal{U}}^{\lambda\dagger}\circ\tilde{\pi}^\dagger_{\lambda\mu}\} = \mathrm{i}\hbar q c^2\{(\partial_\mu\tilde{\mathcal{U}}^\dagger_\lambda)\circ\tilde{\mathcal{U}}^\lambda - \tilde{\mathcal{U}}^\dagger_\lambda\circ(\partial_\mu\tilde{\mathcal{U}}^\lambda)\} \tag{3.23}$$

$$\tilde{\mathcal{M}}^\lambda_{\mu\nu} = \tilde{\mathcal{L}}^\lambda_{\mu\nu} + \tilde{\mathcal{S}}^\lambda_{\mu\nu}\,, \tag{3.24}$$

where the Lorenz conditions were taken into account, q is the charge of the field (of field's particles), and

$$\tilde{\mathcal{L}}^\lambda_{\mu\nu} := x_\mu\,\tilde{\mathcal{T}}^\lambda{}_\nu - x_\nu\,\tilde{\mathcal{T}}^\lambda{}_\mu \tag{3.25}$$

$$\begin{aligned}
\tilde{\mathcal{S}}^\lambda_{\mu\nu} &:= \frac{1}{1+\tau(\tilde{\mathcal{U}})}\{\,\tilde{\pi}^{\rho\lambda}\circ(I^\sigma_{\rho\mu\nu}\tilde{\mathcal{U}}_\sigma) + (I^{\dagger\,\sigma}_{\rho\mu\nu}\tilde{\mathcal{U}}^\dagger_\sigma)\circ\tilde{\pi}^{\rho\lambda\dagger}\} \\
&= \frac{\hbar^2c^2}{1+\tau(\tilde{\mathcal{U}})}\{(\partial^\lambda\tilde{\mathcal{U}}^\dagger_\mu)\circ\tilde{\mathcal{U}}_\nu - (\partial^\lambda\tilde{\mathcal{U}}^\dagger_\nu)\circ\tilde{\mathcal{U}}_\mu - \tilde{\mathcal{U}}^\dagger_\mu\circ(\partial^\lambda\tilde{\mathcal{U}}_\nu) + \tilde{\mathcal{U}}^\dagger_\nu\circ(\partial^\lambda\tilde{\mathcal{U}}_\mu)\}
\end{aligned} \tag{3.26}$$

---

[11] As it is well know [1, 2, 16], in this important case a gauge symmetry arises, i.e. an invariance of the theory under the gauge transformations $\tilde{\mathcal{U}}_\mu \mapsto \tilde{\mathcal{U}}_\mu + \partial_\mu\tilde{\mathcal{K}}$, $\tilde{\mathcal{K}}$ being a C^2 operator, in Heisenberg picture or $\mathcal{U}_\mu \mapsto \tilde{\mathcal{U}}_\mu + \frac{1}{\mathrm{i}\hbar}[K, \mathcal{P}_\mu]$ in momentum picture.

[12] As a consequence of (3.18), the expressions (3.22)–(3.25) are sums of the ones corresponding to $\tilde{\mathcal{U}}_0, \dots, \tilde{\mathcal{U}}_3$ considered as free scalar fields [1, 15, 16]. For a rigorous derivation of (3.22)–(3.26), see the general rules described in [17].

with the numbers

$$I^{\sigma}_{\rho\mu\nu} = I^{\dagger\,\sigma}_{\rho\mu\nu} = \delta^{\sigma}_{\mu}\eta_{\nu\rho} - \delta^{\sigma}_{\nu}\eta_{\mu\rho} \tag{3.27}$$

being characteristics of a vector field under 4-rotations [3, eq. (0-43)]. It should be noticed, since the energy-momentum operator (3.22) is *symmetric*, $\tilde{\mathcal{T}}_{\mu\nu} = \tilde{\mathcal{T}}_{\nu\mu}$, the spin and orbital angular momentum density operators satisfy the continuity equations

$$\partial_\lambda \tilde{\mathcal{S}}^{\lambda}_{\mu\nu} = 0 \quad \partial_\lambda \tilde{\mathcal{L}}^{\lambda}_{\mu\nu} = 0 \tag{3.28}$$

and, consequently, the spin and orbital angular momentum operators of a free vector field are conserved ones, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x^0}\tilde{\mathcal{S}}^{\lambda}_{\mu\nu} = 0 \quad \frac{\mathrm{d}}{\mathrm{d}x^0}\tilde{\mathcal{L}}^{\lambda}_{\mu\nu} = 0. \tag{3.29}$$

According to (2.4), (3.9), (3.12), and (3.18), the densities of the dynamical characteristics of a free vector field in momentum picture are:

$$\begin{aligned}\mathcal{T}_{\mu\nu} = {} & \frac{c^2}{1+\tau(\mathcal{U})}\{[\mathcal{U}^{\dagger}_{\lambda}, \mathcal{P}_{\mu}]_{\_} \circ [\mathcal{U}^{\lambda}, \mathcal{P}_{\nu}]_{\_} + [\mathcal{U}^{\dagger}_{\lambda}, \mathcal{P}_{\nu}]_{\_} \circ [\mathcal{U}^{\lambda}, \mathcal{P}_{\mu}]_{\_}\} \\ & - \frac{\eta_{\mu\nu}c^2}{1+\tau(\mathcal{U})}\{m^2c^2\mathcal{U}^{\dagger}_{\lambda} \circ \mathcal{U}^{\lambda} + [\mathcal{U}^{\dagger}_{\varkappa}, \mathcal{P}_{\lambda}]_{\_} \circ [\mathcal{U}^{\varkappa}, \mathcal{P}^{\lambda}]_{\_}\}\end{aligned} \tag{3.30}$$

$$\mathcal{J}_{\mu} = qc^2\{[\mathcal{U}^{\dagger}_{\lambda}, \mathcal{P}_{\mu}]_{\_} \circ \mathcal{U}^{\lambda} - \mathcal{U}^{\dagger}_{\lambda} \circ [\mathcal{U}^{\lambda}, \mathcal{P}_{\mu}]_{\_}\} \tag{3.31}$$

$$\mathcal{L}^{\lambda}_{\mu\nu} = x_\mu \mathcal{T}^{\lambda}{}_{\nu} - x_\nu \mathcal{T}^{\lambda}{}_{\mu} \tag{3.32}$$

$$\mathcal{S}^{\lambda}_{\mu\nu} = -\frac{\mathrm{i}\hbar c^2}{1+\tau(\mathcal{U})}\{[\mathcal{U}^{\dagger}_{\mu}, \mathcal{P}^{\lambda}]_{\_} \circ \mathcal{U}_{\nu} - [\mathcal{U}^{\dagger}_{\nu}, \mathcal{P}^{\lambda}]_{\_} \circ \mathcal{U}_{\mu} - \mathcal{U}^{\dagger}_{\mu} \circ [\mathcal{U}_{\nu}, \mathcal{P}^{\lambda}]_{\_} + \mathcal{U}^{\dagger}_{\nu} \circ [\mathcal{U}_{\mu}, \mathcal{P}^{\lambda}]_{\_}\}. \tag{3.33}$$

Comparing (3.30) and (3.31) with the corresponding expressions for $\mathcal{U}_0, \dots, \mathcal{U}_3$, considered as free scalar fields [13], we see that the terms originating from $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ enter in (3.30) and (3.31) with right signs if $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ were free scalar fields. But the terms, in which $\mathcal{U}_0$ enters, are with signs opposite to the ones if $\mathcal{U}_0$ was a free scalar field. In particular, this means that the contribution of $\mathcal{U}_0$ in the field's energy is negative. All this points to the known fact that $\mathcal{U}_0$ is a carrier of an unphysical degree of freedom, which must be eliminated (via the Lorenz conditions (3.18)). The second new moment, with respect to the scalar field case, is the existence of a, generally, non-vanishing spin angular momentum density operator (3.33) to which a special attention will be paid (see Sect. 6).

Since $\mathcal{U}_\mu$ are solutions of the Klein-Gordon equations (3.17), the operator $\frac{1}{c^2}[[\cdot, \mathcal{P}_\lambda]_{\_}, \mathcal{P}^\lambda]_{\_}$ has a meaning of a square-of-mass operator of the vector field under consideration. At the same time, the operator $\frac{1}{c^2}\mathcal{P}_\lambda \circ \mathcal{P}^\lambda$ has a meaning of square-of-mass operator of the field's states (state vectors).

We shall specify the relation (2.43) for a vector field by putting $\varepsilon(\mathcal{U}_\mu) = +1$ and $\varepsilon(\mathcal{U}^{\dagger}_{\mu}) = -1$. Therefore the relations (2.43) and (2.44) take the form

$$[\mathcal{U}_\mu, \mathcal{Q}]_{\_} = q\mathcal{U}_\mu \quad [\mathcal{U}^{\dagger}_{\mu}, \mathcal{Q}]_{\_} = -q\mathcal{U}^{\dagger}_{\mu} \tag{3.34}$$

$$[\mathcal{U}_\lambda, \mathcal{M}_{\mu\nu}(x, x_0)]_{\_} = x_\mu[\mathcal{U}_\lambda, \mathcal{P}_\nu]_{\_} - x_\nu[\mathcal{U}_\lambda, \mathcal{P}_\mu]_{\_} + \mathrm{i}\hbar(\mathcal{U}_\mu\eta_{\nu\lambda} - \mathcal{U}_\nu\eta_{\mu\lambda}) \tag{3.35a}$$

$$[\mathcal{U}^{\dagger}_{\lambda}, \mathcal{M}_{\mu\nu}(x, x_0)]_{\_} = x_\mu[\mathcal{U}^{\dagger}_{\lambda}, \mathcal{P}_\nu]_{\_} - x_\nu[\mathcal{U}^{\dagger}_{\lambda}, \mathcal{P}_\mu]_{\_} + \mathrm{i}\hbar(\mathcal{U}^{\dagger}_{\mu}\eta_{\nu\lambda} - \mathcal{U}^{\dagger}_{\nu}\eta_{\mu\lambda}), \tag{3.35b}$$

where (3.27) was used. It is clear, the last terms in (3.35) are due to the spin angular momentum, while the other ones originate from the orbital angular momentum.

4. Analysis of the field equations

The analysis of the Dirac equations in [14] can *mutatis mutandis* be applied to the case of vector fields satisfying the Lorenz condition. This can be done as follows

At first, we distinguish the 'degenerate' solutions

$$[\mathcal{U}, \mathcal{P}_\mu]_{\_} = 0 \quad [\mathcal{U}^\dagger, \mathcal{P}_\mu]_{\_} = 0 \qquad \text{for } m = 0 \tag{4.1}$$

of the Klein-Gordon equations (3.17), which solutions, in view of (2.7), in Heisenberg picture read

$$\tilde{\mathcal{U}}_\mu(x) = \tilde{\mathcal{U}}_\mu(x_0) = \tilde{\mathcal{U}}_\mu \ (= \mathrm{const}_\mu) \quad \tilde{\mathcal{U}}^\dagger_\mu(x) = \tilde{\mathcal{U}}^\dagger_\mu(x_0) = \tilde{\mathcal{U}}^\dagger_\mu \ (= \mathrm{const}^\dagger_\mu) \quad \text{for } m = 0. \tag{4.2}$$

According to equations (3.30)–(3.33), the energy-momentum, charge and angular momentum density operators for the solutions (4.1) respectively are:

$$\mathcal{T}_{\mu\nu} = 0 \quad \mathcal{J}_\mu = 0 \quad \mathcal{L}^\lambda_{\mu\nu} = \mathcal{S}^\lambda_{\mu\nu} = \mathcal{M}^\lambda_{\mu\nu} = 0 \tag{4.3}$$

Since (4.3) and (2.22)–(2.25) imply

$$\mathcal{P}_\mu = 0 \quad \mathcal{Q}_\mu = 0 \quad \mathcal{L}_{\mu\nu} = \mathcal{S}_{\mu\nu} = \mathcal{M}_{\mu\nu} = 0, \tag{4.4}$$

the solutions (4.1) (or (4.2) in Heisenberg picture) describe a massless vector field with vanishing dynamical characteristics. Such a field cannot lead to any predictable observable results and, in this sense is unphysical.[13]

The further analysis of the field equations will be done similarly to the one of free spinor fields in [14]. For the purpose, one should replace the Dirac equations with the Lorenz conditions (3.18) and take into account that now the field equations are (3.17)–(3.18), not only (3.18).[14]

Taking into account the above facts, we can describe the structure of the solutions of the field equations (3.17)–(3.18) as follows.

Proposition 4.1. *The solutions of the equations* (3.17)–(3.18) *and* (2.7) *can be written as (do not sum over μ!)*

$$\mathcal{U}_\mu = \int \mathrm{d}^3\boldsymbol{k} \{ f_{\mu,+}(\boldsymbol{k}) \mathcal{U}_\mu(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} + f_{\mu,-}(\boldsymbol{k}) \mathcal{U}_\mu(k)\big|_{k_0=-\sqrt{m^2c^2+\boldsymbol{k}^2}} \} \tag{4.5a}$$

$$\mathcal{U}^\dagger_\mu = \int \mathrm{d}^3\boldsymbol{k} \{ f^\dagger_{\mu,+}(\boldsymbol{k}) \mathcal{U}^\dagger_\mu(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} + f^\dagger_{\mu,-}(\boldsymbol{k}) \mathcal{U}^\dagger_\mu(k)\big|_{k_0=-\sqrt{m^2c^2+\boldsymbol{k}^2}} \} \tag{4.5b}$$

or, equivalently as

$$\mathcal{U}_\mu = \int \mathrm{d}^4k\delta(k^2 - m^2c^2) f_\mu(k) \mathcal{U}_\mu(k) \quad \mathcal{U}^\dagger_\mu = \int \mathrm{d}^4k\delta(k^2 - m^2c^2) f^\dagger_\mu(k) \mathcal{U}^\dagger_\mu(k). \tag{4.6}$$

Here: $k = (k^0, k^1, k^2, k^3)$ is a 4-vector with dimension of 4-momentum, $k^2 = k_\mu k^\mu = k_0^2 - k_1^2 - k_2 - k_3^2 = k_0^2 - \boldsymbol{k}^2$ with k_μ being the components of k and $\boldsymbol{k} := (k^1, k^2, k^3) = -(k_1, k_2, k_3)$ being

[13] This case is similar to the one of free scalar fields describe in [13]. Note, the so-arising situation is completely different from a similar one, when free spinor Dirac fields are concerned as in it solutions, like (4.1), are in principle observable — see [14].

[14] It is interesting to be noted, the Dirac equation $\mathrm{i}\hbar\gamma^\mu\partial_\mu\tilde{\psi} - mc\tilde{\psi} = 0$, γ^μ being the γ-matrices and $\tilde{\psi}$ a 4-spinor, in the massless case, $m = 0$, takes the form of a Lorenz condition, viz. $\partial_\mu\tilde{\mathcal{U}}^\mu = 0$ with $\tilde{\mathcal{U}}^\mu = \gamma^\mu\tilde{\psi}$.

the 3-dimensional part of k, $\delta(\cdot)$ is the (1-dimensional) Dirac delta function, the operators $\mathcal{U}_\mu(k), \mathcal{U}^\dagger_\mu(k)\colon \mathcal{F}\to\mathcal{F}$ are solutions of the equations

$$[\mathcal{U}_\mu(k), \mathcal{P}_\nu]_{\_} = -k_\nu \mathcal{U}_\mu(k) \qquad [\mathcal{U}^\dagger_\mu(k), \mathcal{P}_\nu]_{\_} = -k_\nu \mathcal{U}^\dagger_\mu(k) \tag{4.7a}$$

$$\{k^\mu \mathcal{U}_\mu(k)\}\big|_{k^2=m^2c^2} = 0 \qquad \{k^\mu \mathcal{U}^\dagger_\mu(k)\}\big|_{k^2=m^2c^2} = 0, \tag{4.7b}$$

$f_{\mu,\pm}(\boldsymbol{k})$ and $f^\dagger_{\mu,\pm}(\boldsymbol{k})$ are complex-valued functions (resp. distributions (generalized functions)) of $\boldsymbol{k}$ for solutions different from (4.1) *(resp. for the solutions* (4.1)*), and f_μ and $f^\dagger_\mu$ are complex-valued functions (resp. distribution) of k for solutions different from* (4.1) *(resp. for the solutions* (4.1)*). Besides, we have the relations $f_\mu(k)|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}} = 2\sqrt{m^2c^2+\boldsymbol{k}^2}\, f_{\mu,\pm}(\boldsymbol{k})$ and $f^\dagger_\mu(k)|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}} = 2\sqrt{m^2c^2+\boldsymbol{k}^2}\, f^\dagger_{\mu,\pm}(\boldsymbol{k})$ for solutions different from* (4.1)*.*

Remark 4.1. Evidently, in (4.5) and (4.6) enter only the solutions of (4.7) for which

$$k^2 := k_\mu k^\mu = k_0^2 - \boldsymbol{k}^2 = m^2c^2. \tag{4.8}$$

This circumstance is a consequence of the fact that $\mathcal{U}_\mu$ the solutions of the Klein-Gordon equations (3.17).

Remark 4.2. Obviously, to the solutions (4.1) corresponds (4.7a) with $\mathcal{P}_\mu = 0$. Hence

$$\tilde{\mathcal{U}}_\mu(x,0) = \mathcal{U}_\mu(0) = \text{const} \quad \tilde{\mathcal{U}}^\dagger_\mu(x,0) = \mathcal{U}^\dagger_\mu(0) = \text{const} \quad \mathcal{P}_\mu = \tilde{\mathcal{P}}_\mu = 0 \tag{4.9}$$

with (see (2.4))

$$\tilde{\mathcal{U}}_\mu(x,k) := \mathcal{U}^{-1}(x,x_0)\circ\mathcal{U}_\mu(k)\circ\mathcal{U}(x,x_0) \quad \tilde{\mathcal{U}}^\dagger_\mu(x,k) := \mathcal{U}^{-1}(x,x_0)\circ\mathcal{U}^\dagger_\mu(k)\circ\mathcal{U}(x,x_0). \tag{4.10}$$

These solutions, in terms of (4.5) or (4.6), are described by $m = 0$ and, for example, $f_{\mu,\pm}(\boldsymbol{k}) = f^\dagger_{\mu,\pm}(\boldsymbol{k}) = (\frac{1}{2}\pm a)\delta^3(\boldsymbol{k})$ for some $a\in\mathbb{C}$ or $f_\mu(k) = f^\dagger_\mu(k)$ such that $f_\mu(k)|_{k_0=\pm|\boldsymbol{k}|} = (1\pm 2a)|\boldsymbol{k}|\delta^3(\boldsymbol{k})$, respectively. (Here $\delta^3(\boldsymbol{k}) := \delta(k^1)\delta(k^2)\delta(k^3)$ is the 3-dimensional Dirac delta-function. Note the equality $\delta(y^2-b^2) = \frac{1}{b}(\delta(y+b)+\delta(y-b))$ for $b>0$.)

Remark 4.3. Since $\mathcal{U}^\dagger_\mu := (\mathcal{U}_\mu)^\dagger$, from (4.5) (resp. (4.6)) is clear that there should exist some connection between $f_{\mu,\pm}(\boldsymbol{k})\mathcal{U}_\mu(k)$ and $f^\dagger_{\mu,\pm}(\boldsymbol{k})\mathcal{U}^\dagger_\mu(k)$ with $k_0 = +\sqrt{m^2c^2+\boldsymbol{k}^2}$ (resp. between $f_\mu(k)\mathcal{U}_\mu(k)$ and $f^\dagger_\mu(k)\mathcal{U}^\dagger_\mu(k)$). A simple examination of (4.5) (resp. (4.6)) reveals that the Hermitian conjugation can either transform these expressions into each other or 'change' the signs plus and minus in them according to:

$$\big(f_{\mu\pm}(\boldsymbol{k})\mathcal{U}_\mu(k)\big|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}}\big)^\dagger = -f^\dagger_{\mu,\mp}(-\boldsymbol{k})\mathcal{U}^\dagger_\mu(-k)\big|_{k_0=\mp\sqrt{m^2c^2+\boldsymbol{k}^2}}. \tag{4.11a}$$

$$\big(f^\dagger_{\mu,\pm}(\boldsymbol{k})\mathcal{U}^\dagger_\mu(k)\big|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}}\big)^\dagger = -f_{\mu,\mp}(-\boldsymbol{k})\mathcal{U}_\mu(-k)\big|_{k_0=\mp\sqrt{m^2c^2+\boldsymbol{k}^2}} \tag{4.11b}$$

$$\big(f_\mu(k)\mathcal{U}_\mu(k)\big)^\dagger = f^\dagger_\mu(-k)\mathcal{U}^\dagger_\mu(-k) \tag{4.12a}$$

$$\big(f^\dagger_\mu(k)\mathcal{U}^\dagger_\mu(k)\big)^\dagger = f(-k)\mathcal{U}_\mu(-k). \tag{4.12b}$$

From the below presented proof of proposition 4.1 and the comments after it, it will be clear that (4.11) and (4.12) should be accepted. Notice, the above equations mean that $\mathcal{U}^\dagger_\mu(k)$ is *not* the Hermitian conjugate of $\mathcal{U}_\mu(k)$.

Proof. The proposition was proved for the solutions (4.1) in remark 4.2. So, below we suppose that $(k,m)\neq(0,0)$.

The equivalence of (4.5) and (4.6) follows from $\delta(y^2-b^2)=\frac{1}{b}(\delta(y+b)+\delta(y-b))$ for $b>0$.

Since $\mathcal{U}_\mu$ and $\mathcal{U}^\dagger_\mu$ are solutions of the Klein-Gordon equations (3.17), the representations (4.5) and the equalities (4.11) and (4.12), with $\mathcal{U}_\mu(k)$ and $\mathcal{U}^\dagger_\mu(k)$ satisfying (4.7a), follow from the proved in [13] similar proposition 4.1 describing the structure of the solutions of the Klein-Gordon equation in momentum picture.[15]

At the end, inserting (4.5) or (4.6) into (3.18), we obtain the equations (4.7b), due to (4.7a). □

From the proof of proposition 4.1, as well as from the one of [13, proposition 4.1] the next two conclusions can be made. On one hand, the conditions (4.7a) ensure that (4.5) and (4.6) are solutions of (2.7) and the Klein-Gordon equations (3.17), while (4.7b) single out between them the ones satisfying the Lorenz conditions (3.18). On other hand, since up to a phase factor and, possibly, normalization constant, the expressions $f_\mu(k)\mathcal{U}_\mu(k)$ and $f^\dagger_\mu(k)\mathcal{U}_\mu(k)$ coincide with the Fourier images of respectively $\tilde{\mathcal{U}}_\mu(x)$ and $\tilde{\mathcal{U}}^\dagger_\mu(x)$ in Heisenberg picture, we can write

$$\mathcal{U}_\mu=\int\delta(k^2-m^2c^2)\underline{\mathcal{U}_\mu}(k)\,\mathrm{d}^4k\quad \tilde{\mathcal{U}}_\mu(x)=\int\delta(k^2-m^2c^2)\underline{\mathcal{U}_\mu}(k)\mathrm{e}^{\mathrm{i}\frac{1}{\hbar}(x^\mu-x^\mu_0)k_\mu}\,\mathrm{d}^4k \tag{4.13}$$

and similarly for $\mathcal{U}^\dagger_\mu$ (with $\big(\underline{\mathcal{U}^\dagger_\mu}(k)\big)^\dagger=\underline{\mathcal{U}^\dagger_\mu}(-k)$), where $\underline{\mathcal{U}_\mu}(k)$ are suitably normalized solutions of (4.7). Therefore, up to normalization factor, the Fourier images of $\tilde{\mathcal{U}}_\mu(x)$ and $\mathcal{U}^\dagger_\mu(x)$ are

$$\underline{\tilde{\mathcal{U}}_\mu}(k)=\mathrm{e}^{\frac{1}{\mathrm{i}\hbar}x^\mu_0k_\mu}\underline{\mathcal{U}_\mu}(k)\qquad \underline{\tilde{\mathcal{U}}^\dagger_\mu}(k)=\mathrm{e}^{\frac{1}{\mathrm{i}\hbar}x^\mu_0k_\mu}\underline{\mathcal{U}^\dagger_\mu}(k) \tag{4.14}$$

where x_0 is a fixed point (see Sect. 2). So, the momentum representation of free vector field (satisfying the Lorenz condition) in Heisenberg picture is an appropriately chosen operator base for the solutions of the equations (3.17)–(3.18) and (2.7) in momentum picture. This conclusion allows us freely to apply in momentum picture the existing results concerning that basis in Heisenberg picture.

As anyone of the equations (4.7b) is a linear homogeneous equation with respect to $\mathcal{U}_\mu$ and $\mathcal{U}^\dagger_\mu$, each of these equations has exactly *three linearly independent* solutions, which will be labeled by indices $s,s',t,\dots$ taking the values 1, 2 and 3, $s,s',t=1,2,3$.[16] Define the operator-valued vectors $\mathcal{U}^\mu_{s,(\pm)}(k)$ and $\mathcal{U}^{\mu\dagger}_{s,(\pm)}(k)$, where $s=1,2,3$ and the index $(\pm)$ indicates the sign of $k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}$ in (4.7b), as linearly independent solutions of the equations

$$k_\mu\big|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}^\mu_{s,(\pm)}(k)=0\qquad k_\mu\big|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}^{\mu\dagger}_{s,(\pm)}(k)=0. \tag{4.15}$$

As a consequence of (4.7a), they also satisfy the relations

$$[\mathcal{U}^\mu_{s,(\pm)}(k),\,\mathcal{P}_\mu]_{\_}=-k_\nu\,\mathcal{U}^\mu_{s,(\pm)}(k)\qquad [\mathcal{U}^{\mu\dagger}_{s,(\pm)}(k),\,\mathcal{P}_\mu]_{\_}=-k_\nu\,\mathcal{U}^{\mu\dagger}_{s,(\pm)}(k). \tag{4.16}$$

Since any solution of the first (resp. second) equation in (4.7b) can be represented as a linear combination of $\mathcal{U}^\mu_{s,(\pm)}(k)$ and $\mathcal{U}^{\mu\dagger}_{s,(\pm)}(k)$, $s=1,2,3$, we can rewrite (4.5) as (do not sum over

[15] One can prove the representations (4.5), under the conditions (4.7), by repeating *mutatis mutandis* the proof of [13, proposition 4.1]. From it the equalities (4.11) and (4.12) rigorously follow too.

[16] These indices, which will be referred as the *polarization* or *spin* indices, have nothing common with the spacial indices $a,b,\dots=1,2,3$ labeling the spacial components of 4-vectors or tensors.

μ!)

$$\mathcal{U}_\mu = \sum_s \int \mathrm{d}^3\boldsymbol{k}\{f_{\mu,s,+}(\boldsymbol{k})\mathcal{U}_{\mu,s,+}(k) + f_{\mu,s,-}(\boldsymbol{k})\mathcal{U}_{\mu,s,-}(k)\}\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} \tag{4.17a}$$

$$\mathcal{U}^\dagger_\mu = \sum_s \int \mathrm{d}^3\boldsymbol{k}\{f^\dagger_{\mu,s,+}(\boldsymbol{k})\mathcal{U}^\dagger_{\mu,s,+}(k) + f^\dagger_{\mu,s,-}(\boldsymbol{k})\mathcal{U}^\dagger_{\mu,s,-}(k)\}\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}}, \tag{4.17b}$$

where $f_{\mu,s,\pm}(\boldsymbol{k})$ and $f^\dagger_{\mu,s,\pm}(\boldsymbol{k})$ are some complex-valued (generalized) functions of $\boldsymbol{k}$ such that

$$\begin{aligned}
\sum_s f_{\mu,s,\pm}(\boldsymbol{k})\mathcal{U}_{\mu,s,\pm}(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} &= f_{\mu,\pm}(\boldsymbol{k})\mathcal{U}_\mu(k)\big|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}}\\
\sum_s f^\dagger_{\mu,s,\pm}(\boldsymbol{k})\mathcal{U}^\dagger_{\mu,s,\pm}(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} &= f^\dagger_{\mu,\pm}(\boldsymbol{k})\mathcal{U}^\dagger_\mu(k)\big|_{k_0=\pm\sqrt{m^2c^2+\boldsymbol{k}^2}}.
\end{aligned} \tag{4.18}$$

In what follows, we shall need a system of *classical*, not operator-valued, suitably normalized solutions of the equations (4.7b), which equations reflect the Lorenz conditions (3.18), The idea of their introduction lies in the separation of the frame-independent properties of a free vector field from the particular representation of that field in a particular frame of reference. It will be realized below in Sect. 5.

Consider the equation

$$k^\mu\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} v_\mu(\boldsymbol{k}) = 0 \tag{4.19}$$

where $v_\mu(\boldsymbol{k})$ is a *classical* 4-vector field (over the $\boldsymbol{k}$-space). This is a *single* linear and homogeneous equation with respect to *four* functions $v_\mu(\boldsymbol{k})$, $\mu = 0,1,2,3$. Therefore (4.19) admits *three* linearly independent solutions. Define $v^s_\mu(\boldsymbol{k})$, with $s = 1,2,3$ and $\mu = 0,1,2,3$, as linearly independent solutions of

$$k^\mu\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} v^s_\mu(\boldsymbol{k}) = 0 \tag{4.20}$$

satisfying the conditions

$$v^s_\mu(\boldsymbol{k})v^{\mu,s'}(\boldsymbol{k}) = -\delta^{ss'}(1-\delta_{0m}\delta_{s3}) = -\delta^{ss'}\times\begin{cases}1 & \text{for } m\neq 0\\ \delta^{1s}+\delta^{2s} & \text{for } m=0\end{cases} \tag{4.21}$$

where $v^{\mu,s}(\boldsymbol{k}) := \eta^{\mu\nu}v^s_\nu(\boldsymbol{k})$. In more details, the relations (4.21) read

$$v^s_\mu(\boldsymbol{k})v^{\mu,s'}(\boldsymbol{k}) = -\delta^{ss'} \qquad \text{for } m\neq 0 \tag{4.22a}$$

$$\{v^s_\mu(\boldsymbol{k})v^{\mu,s'}(\boldsymbol{k})\}\big|_{m=0} = \begin{cases}-\delta^{ss'} & \text{if } (s,s')\neq(3,3)\\ 0 & \text{if } (s,s')=(3,3)\end{cases} = \begin{cases}-\delta^{ss'} & \text{if } s,s'=1,2\\ 0 & \text{otherwise}\end{cases} \tag{4.22b}$$

The reader may verify by a direct calculation, an explicit solution of (4.20)–(4.21), for $\boldsymbol{k}\neq\boldsymbol{0}$, is provided by

$$\begin{aligned}
&v^1_a(\boldsymbol{k}) = e^1_a(\boldsymbol{k}) \quad v^2_a(\boldsymbol{k}) = e^2_a(\boldsymbol{k})\\
&v^3_a(\boldsymbol{k}) = \frac{k_a}{\sqrt{\boldsymbol{k}^2}}\Big(\frac{m^2c^2+\boldsymbol{k}^2}{m^2c^2+\boldsymbol{k}^2\delta_{0m}}\Big)^{1/2} = \frac{k_a}{\sqrt{\boldsymbol{k}^2}}\sqrt{m^2c^2+\boldsymbol{k}^2}\times\begin{cases}\frac{1}{mc} & \text{for } m\neq 0\\ \frac{1}{\sqrt{\boldsymbol{k}^2}} & \text{for } m=0\end{cases}\\
&v^s_0(\boldsymbol{k}) = -\frac{1}{\sqrt{m^2c^2+\boldsymbol{k}^2}}\sum_{a=1}^3 k^a v^s_a(\boldsymbol{k}) = \Big(\frac{\boldsymbol{k}^2}{m^2c^2+\boldsymbol{k}^2\delta_{0m}}\Big)^{1/2}\delta^{3s}
\end{aligned} \tag{4.23}$$

where the vectors $e^1_a(\boldsymbol{k})$ and $e^2_a(\boldsymbol{k})$ are such that

$$\boldsymbol{e}^s(\boldsymbol{k})\cdot\boldsymbol{e}^{s'}(\boldsymbol{k})=\sum_a e^s_a(\boldsymbol{k})e^{s'}_a(\boldsymbol{k})=\delta^{ss'}\quad \boldsymbol{e}^s(\boldsymbol{k})\cdot\boldsymbol{v}^3(\boldsymbol{k})=\sum_a e^s_a(\boldsymbol{k})v^3_a(\boldsymbol{k})=0\quad\text{for } s,s'=1,2, \tag{4.24}$$

i.e. the 3-vectors $\boldsymbol{v}^1(\boldsymbol{k})$, $\boldsymbol{v}^2(\boldsymbol{k})$ and $\boldsymbol{v}^3(\boldsymbol{k})$ form an orthogonal (orthonormal for $m=0$) basis in the $\mathbb{R}^3$ $\boldsymbol{k}$-space with $\boldsymbol{v}^3(\boldsymbol{k})$ being proportional to (having the direction of) $\boldsymbol{k}$. Here $\delta_{0m}:=0$ for $m\neq 0$ and $\delta_{0m}:=1$ for $m=0$. If $\boldsymbol{k}=\boldsymbol{0}$, one can put

$$\begin{aligned}
&v^s_0(\boldsymbol{0})=0\quad v^1_a(\boldsymbol{0})=e^1_a(\boldsymbol{0})\quad v^2_a(\boldsymbol{0})=e^2_a(\boldsymbol{0})\\
&v^3_a(\boldsymbol{0})=-(1-\delta_{0m})\sum_{bc}\varepsilon^{abc}e^1_b(\boldsymbol{0})e^2_c(\boldsymbol{0})=\begin{cases}-\varepsilon^{abc}e^1_b(\boldsymbol{0})e^2_c(\boldsymbol{0}) & \text{for } m\neq 0\\ 0 & \text{for } m=0\end{cases}
\end{aligned} \tag{4.25}$$

with $e^1_a(\boldsymbol{0})$ and $e^2_a(\boldsymbol{0})$ satisfying (4.24) with $\boldsymbol{k}=0$; in particular, one can put $e^1_a(\boldsymbol{0})=-\delta^1_a$ and $e^2_a(\boldsymbol{0})=-\delta^2_a$.[17]

The solutions (4.23) and (4.25) of (4.20)–(4.21) satisfy the following relations for summation with respect to the polarization index ($k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}$):

$$\sum_{s=1}^{3}\{v^s_\mu(\boldsymbol{k})v^s_\nu(\boldsymbol{k})\}\big|_{\boldsymbol{k}\neq\boldsymbol{0}}=-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{m^2c^2+\boldsymbol{k}^2\delta_{0m}}\times\begin{cases}1+\delta_{0m} & \text{for } \mu=\nu=0\\ 1-\delta_{0m} & \text{for } \mu=\nu=1,2,3\\ 1 & \text{otherwise}\end{cases} \tag{4.26a}$$

$$\sum_{s=1}^{3}v^s_\mu(\boldsymbol{0})v^s_\nu(\boldsymbol{0})=\begin{cases}\delta_{\mu\nu} & \text{if } m\neq 0 \text{ and } \mu,\nu=1,2,3 \text{ or if } m=0 \text{ and } \mu,\nu=1,2\\ 0 & \text{otherwise}\end{cases} \tag{4.26b}$$

$$\sum_{s=1,2}v^s_\mu(\boldsymbol{k})v^s_\nu(\boldsymbol{k})=\begin{cases}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2\\ 0 & \text{otherwise}\end{cases}\quad . \tag{4.27}$$

These equations are simple corollaries of (4.14)–(4.25) in a 3-frame specified by the unit vectors defined by $v^s_a(\boldsymbol{k})$, $s=1,2,3$. If we introduce the vectors $\eta^\mu:=(1,0,0,0)$ and $\hat{k}^\mu:=\big(0,\frac{\boldsymbol{k}}{\sqrt{\boldsymbol{k}^2}}\big)=\frac{k^\mu-(k\eta)\eta^\mu}{\sqrt{(k\eta)-k^2}}$ with $(k\eta):=k_\mu\eta^\mu$ and $\hat{k}^\mu\big|_{\boldsymbol{k}=\boldsymbol{0}}:=(0,0,0,1)$, then the covariant form of (4.27) is

$$\sum_{s=1,2}v^s_\mu(\boldsymbol{k})v^s_\nu(\boldsymbol{k})=-\eta_{\mu\nu}+\eta_\mu\eta_\nu-\hat{k}_\mu\hat{k}_\nu, \tag{4.28}$$

which, for $m=0$, coincides with [15, eq. (14.53)], where $v^s_\mu(\boldsymbol{k})|_{m=0}$ with $s=1,2$ is denoted by $\varepsilon_\mu(k,s)$, but it is supposed that $\varepsilon_0(k,s):=0$ and the value $s=3$ is excluded by definition. Notice, the last multiplier in (4.26a) can be written in a covariant form as $(1+\eta_{\mu\nu}\delta_{0m})$. The easiest way for proving these equalities for $\mu,\nu=1,2,3$ is in a frame in which $k^1=k^2=0$. The rest of the equations are consequences of the ones with $\mu,\nu=1,2,3$, (4.23) and (4.25).

5. Frequency decompositions and creation and annihilation operators

The frequency decompositions of a free vector field, satisfying the Lorenz condition, can be introduced similarly to the ones of a free scalar field [13], if (4.5) is used, or of a free spinor

[17] The expression for $v^3_a(\boldsymbol{k})$ in (4.23) is not defined for $\boldsymbol{k}=\boldsymbol{0}$. Indeed, the limit of $k_a/\sqrt{\boldsymbol{k}^2}$, when $\boldsymbol{k}\to\boldsymbol{0}$, depends on how $\boldsymbol{k}$ approaches to the zero vector $\boldsymbol{0}$; one can force $k_a/\sqrt{\boldsymbol{k}^2}$ to tends to any real number by an appropriate choice of the limiting process $\boldsymbol{k}\to\boldsymbol{0}$. For instance $\lim_{\alpha\to 0}\frac{k_a}{\sqrt{\boldsymbol{k}^2}}=-\frac{\delta_{1a}+\delta_{2a}\beta}{\sqrt{1+\beta^2}}$ if $\boldsymbol{k}=\alpha(1,\beta,0)$ for some $\beta\in\mathbb{R}$.

field [14], if (4.17) is used, i.e. if the spin of the field is taken into account. Respectively, we put:

$$\mathcal{U}_\mu^\pm(k) := \begin{cases} f_{\mu,\pm}(\pm\boldsymbol{k})\mathcal{U}_\mu(\pm k) & \text{for } k_0 \ge 0 \\ 0 & \text{for } k_0 < 0 \end{cases} \qquad \mathcal{U}_\mu^{\dagger\pm}(k) := \begin{cases} f_{\mu,\pm}^\dagger(\pm\boldsymbol{k})\mathcal{U}_\mu^\dagger(\pm k) & \text{for } k_0 \ge 0 \\ 0 & \text{for } k_0 < 0 \end{cases} \tag{5.1}$$

$$\begin{aligned} \mathcal{U}_{\mu,s}^\pm(k) &:= \begin{cases} f_{\mu,s,\pm}(\pm\boldsymbol{k})\mathcal{U}_{\mu,s,(\pm)}(\pm k) & \text{for } k_0 \ge 0 \\ 0 & \text{for } k_0 < 0 \end{cases} \\ \mathcal{U}_{\mu,s}^{\dagger\pm}(k) &:= \begin{cases} f_{\mu,s,\pm}^\dagger(\pm\boldsymbol{k})\mathcal{U}_{\mu,s,(\pm)}^\dagger(\pm k) & \text{for } k_0 \ge 0 \\ 0 & \text{for } k_0 < 0 \end{cases}, \end{aligned} \tag{5.2}$$

where $k^2 = m^2c^2$ and $s = 1, 2, 3$. As a consequence of (4.18), we have

$$\mathcal{U}_\mu^\pm(k) = \sum_{s=1}^{3} \mathcal{U}_{\mu,s}^\pm(k) \quad \mathcal{U}_\mu^{\dagger\pm}(k) = \sum_{s=1}^{3} \mathcal{U}_{\mu,s}^{\dagger\pm}(k). \tag{5.3}$$

These operators satisfy the equations

$$\big(\mathcal{U}_\mu^\pm(k)\big)^\dagger = \mathcal{U}_\mu^{\dagger\mp}(k) \quad \big(\mathcal{U}_\mu^{\dagger\pm}(k)\big)^\dagger = \mathcal{U}_\mu^{\mp}(k) \tag{5.4}$$

due to (4.11).

It will be convenient for the following the definitions (5.1) and (5.2) to be specified when $k_0 = +\sqrt{m^2c^2 + \boldsymbol{k}^2} \ge 0$:

$$\begin{aligned} \mathcal{U}_\mu^\pm(\boldsymbol{k}) &:= \mathcal{U}_\mu^\pm(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} = \sum_{s=1}^{3} \mathcal{U}_{\mu,s}^\pm(\boldsymbol{k}) \quad \mathcal{U}_{\mu,s}^\pm(\boldsymbol{k}) := \mathcal{U}_{\mu,s}^\pm(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} \\ \mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k}) &:= \mathcal{U}_\mu^{\dagger\pm}(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}} = \sum_{s=1}^{3} \mathcal{U}_{\mu,s}^{\dagger\pm}(\boldsymbol{k}) \quad \mathcal{U}_{\mu,s}^{\dagger\pm}(\boldsymbol{k}) := \mathcal{U}_{\mu,s}^{\dagger\pm}(k)\big|_{k_0=+\sqrt{m^2c^2+\boldsymbol{k}^2}}. \end{aligned} \tag{5.5}$$

Combining (5.1)–(5.3), (4.5), (4.7) and (4.17), we get

$$\mathcal{U}_\mu = \mathcal{U}_\mu^+ + \mathcal{U}_\mu^- \qquad \mathcal{U}_\mu^\dagger = \mathcal{U}_\mu^{\dagger+} + \mathcal{U}_\mu^{\dagger-} \tag{5.6}$$

$$\mathcal{U}_\mu^\pm := \sum_s \int \mathrm{d}^3\boldsymbol{k}\,\mathcal{U}_{\mu,s}^\pm(\boldsymbol{k}) = \int \mathrm{d}^3\boldsymbol{k}\,\mathcal{U}_\mu^\pm(\boldsymbol{k}) \quad \mathcal{U}_\mu^{\dagger\pm} := \sum_s \int \mathrm{d}^3\boldsymbol{k}\,\mathcal{U}_{\mu,s}^{\dagger\pm}(\boldsymbol{k}) = \int \mathrm{d}^3\boldsymbol{k}\,\mathcal{U}_\mu^\pm(\boldsymbol{k}) \tag{5.7}$$

$$[\mathcal{U}_\mu^\pm(\boldsymbol{k}),\, \mathcal{P}_\nu]_{\_} = \mp k_\nu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}_\mu^\pm(\boldsymbol{k}) \tag{5.8a}$$

$$[\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k}),\, \mathcal{P}_\nu]_{\_} = \mp k_\nu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k}). \tag{5.8b}$$

The equations (4.7b) are incorporated in the above equalities via (5.2), (5.3) and (4.17).

The physical meaning of the above-introduced operators is a consequence of (5.8) and the equations

$$[\mathcal{U}_\mu^\pm(\boldsymbol{k}),\, \mathcal{Q}]_{\_} = q\,\mathcal{U}_\mu^\pm(\boldsymbol{k}) \qquad [\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k}),\, \mathcal{Q}]_{\_} = -q\,\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k}) \tag{5.9}$$

$$[\mathcal{U}_\lambda^\pm(\boldsymbol{k}),\,\mathcal{M}_{\mu\nu}(x)]_- = \{\mp(x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\delta_\lambda^\sigma + \mathrm{i}\hbar(\delta_\mu^\sigma\eta_{\nu\lambda} - \delta_\nu^\sigma\eta_{\mu\lambda})\}\,\mathcal{U}_\sigma^\pm(\boldsymbol{k}) \quad (5.10a)$$

$$[\mathcal{U}_\lambda^{\dagger\pm}(\boldsymbol{k}),\,\mathcal{M}_{\mu\nu}(x)]_- = \{\mp(x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\delta_\lambda^\sigma + \mathrm{i}\hbar(\delta_\mu^\sigma\eta_{\nu\lambda} - \delta_\nu^\sigma\eta_{\mu\lambda})\}\,\mathcal{U}_\sigma^{\dagger\pm}(\boldsymbol{k}), \quad (5.10b)$$

which follow from (3.34) and (3.35). Recall, these equations are external to the Lagrangian formalism, but, in general, they agree with the particle interpretation of the theory. Therefore the below-presented results, in particular the physical interpretation of the creation and annihilation operators, should be accepted with some reserve. However, after the establishment of the particle interpretation of the theory (see Sect 10), the results of this section will be confirmed.

Let $\mathcal{X}_p$, $\mathcal{X}_e$ and $\mathcal{X}_m$ denote state vectors of a vector field with fixed respectively 4-momentum p_μ, (total) charge e and (total) angular momentum $m_{\mu\nu}(x)$, i.e.

$$\mathcal{P}_\mu(\mathcal{X}_p) = p_\mu\,\mathcal{X}_p \quad (5.11a)$$
$$\mathcal{Q}(\mathcal{X}_e) = e\,\mathcal{X}_e \quad (5.11b)$$
$$\mathcal{M}_{\mu\nu}(x)(\mathcal{X}_m) = m_{\mu\nu}(x)\,\mathcal{X}_m. \quad (5.11c)$$

Combining these equations with (5.6)–(5.10), we obtain

$$\mathcal{P}_\mu\big(\mathcal{U}_\mu^\pm(\boldsymbol{k})(\mathcal{X}_p)\big) = (p_\mu \pm k_\mu)\,\mathcal{U}_\mu^\pm(\boldsymbol{k})(\mathcal{X}_p) \qquad k_0 = \sqrt{m^2c^2+\boldsymbol{k}^2} \quad (5.12a)$$
$$\mathcal{P}_\mu\big(\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_p)\big) = (p_\mu \pm k_\mu)\,\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_p) \qquad k_0 = \sqrt{m^2c^2+\boldsymbol{k}^2} \quad (5.12b)$$

$$\mathcal{Q}\big(\mathcal{U}_\mu(\mathcal{X}_e)\big) = (e-q)\,\mathcal{U}_\mu(\mathcal{X}_e) \quad \mathcal{Q}\big(\mathcal{U}_\mu^\dagger(\mathcal{X}_e)\big) = (e+q)\,\mathcal{U}_\mu^\dagger(\mathcal{X}_e) \quad (5.13a)$$
$$\mathcal{Q}\big(\mathcal{U}_\mu^\pm(\mathcal{X}_e)\big) = (e-q)\,\mathcal{U}_\mu^\pm(\mathcal{X}_e) \quad \mathcal{Q}\big(\mathcal{U}_\mu^{\dagger\pm}(\mathcal{X}_e)\big) = (e+q)\,\mathcal{U}_\mu^{\dagger\pm}(\mathcal{X}_e) \quad (5.13b)$$
$$\mathcal{Q}\big(\mathcal{U}_\mu^\pm(\boldsymbol{k})(\mathcal{X}_e)\big) = (e-q)\,\mathcal{U}_\mu^\pm(\boldsymbol{k})(\mathcal{X}_e) \quad \mathcal{Q}\big(\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_e)\big) = (e+q)\,\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_e) \quad (5.13c)$$

$$\begin{aligned}\mathcal{M}_{\mu\nu}(x)\big(\mathcal{U}_\lambda^\pm(\boldsymbol{k})(\mathcal{X}_m)\big) = \{&(m_{\mu\nu}(x) \pm (x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}})\delta_\lambda^\sigma \\ &- \mathrm{i}\hbar(\delta_\mu^\sigma\eta_{\nu\lambda} - \delta_\nu^\sigma\eta_{\mu\lambda})\}\,\mathcal{U}_\sigma^\pm(\boldsymbol{k})(\mathcal{X}_m)\end{aligned} \quad (5.14a)$$
$$\begin{aligned}\mathcal{M}_{\mu\nu}(x)\big(\mathcal{U}_\lambda^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_m)\big) = \{&(m_{\mu\nu}(x) \pm (x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}})\delta_\lambda^\sigma \\ &- \mathrm{i}\hbar(\delta_\mu^\sigma\eta_{\nu\lambda} - \delta_\nu^\sigma\eta_{\mu\lambda})\}\,\mathcal{U}_\sigma^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_m).\end{aligned} \quad (5.14b)$$

The equations (5.12) (resp. (5.13)) show that the eigenvectors of the momentum (resp. charge) operator are mapped into such vectors by the operators $\mathcal{U}_\mu^\pm(\boldsymbol{k})$ and $\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})$ (resp. $\mathcal{U}_\mu$, $\mathcal{U}_\mu^\pm$, $\mathcal{U}_\mu^\pm(\boldsymbol{k})$, $\mathcal{U}_\mu^\dagger$, $\mathcal{U}_\mu^{\dagger\pm}$, and $\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})$). However, by virtue of the equalities (5.14), no one of the operators $\mathcal{U}_\mu$, $\mathcal{U}_\mu^\pm$, $\mathcal{U}_\mu^\pm(\boldsymbol{k})$, $\mathcal{U}_\mu^\dagger$, $\mathcal{U}_\mu^{\dagger\pm}$, and $\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})$ maps an eigenvector of the angular momentum operator into such a vector. The cause for this fact are the matrices $I_{\mu\nu} := [\delta_\mu^\sigma\eta_{\nu\lambda} - \delta_\nu^\sigma\eta_{\mu\lambda}]_{\lambda,\sigma=0}^3$ appearing in (5.14), which generally are non-diagonal and, consequently mix the components of the vectors $\mathcal{U}_\mu(\mathcal{X}_m)$, $\mathcal{U}_\mu^\pm(\mathcal{X}_m)$, $\mathcal{U}_\mu^\pm(\boldsymbol{k})(\mathcal{X}_m)$, $\mathcal{U}_\mu^\dagger(\mathcal{X}_m)$, $\mathcal{U}_\mu^{\dagger\pm}(\mathcal{X}_m)$, and $\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_m)$ in (5.14). Since the matrices $\pm\mathrm{i}\hbar I_{\mu\nu}$ have a dimension of angular momentum and, obviously, originate from the 'pure spin' properties of vector fields, we shall refer to them as *spin-mixing angular momentum matrices* or simply as *spin-mixing matrices*; by definition, the spin-mixing matrix of the field $\mathcal{U}_\mu$ and its Hermitian conjugate $\mathcal{U}_\mu^\dagger$ is $-\mathrm{i}\hbar I_{\mu\nu}$. More generally, if $\mathcal{X}$ is a state vector and $\mathcal{M}_{\mu\nu}(x)\big(\mathcal{U}_\lambda^\pm(\boldsymbol{k})(\mathcal{X})\big) = \{l_{\mu\nu}(x)\delta_\lambda^\sigma + s_{\lambda\mu\nu}^\sigma\}\,\mathcal{U}_\sigma^\pm(\boldsymbol{k})(\mathcal{X})$ or $\mathcal{M}_{\mu\nu}(x)\big(\mathcal{U}_\lambda^{\dagger\pm}(\boldsymbol{k})(\mathcal{X})\big) = \{l_{\mu\nu}^\dagger(x)\delta_\lambda^\sigma + s_{\lambda\mu\nu}^{\dagger\,\sigma}\}\,\mathcal{U}_\sigma^{\dagger\pm}(\boldsymbol{k})(\mathcal{X})$ where $l_{\mu\nu}$ and $l_{\mu\nu}^\dagger$ are some operators and $s_{\mu\nu} := [s_{\lambda\mu\nu}^\sigma]_{\lambda,\sigma=0}^3$ and $s_{\mu\nu}^\dagger := [s_{\lambda\mu\nu}^{\dagger\,\sigma}]_{\lambda,\sigma=0}^3$ are matrices, not proportional to the unit matrix $\mathbb{1}_4$, with operator entries, then we shall say that the operators $\mathcal{U}_\mu^\pm(\boldsymbol{k})$ or $\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})$ have,

respectively, spin-mixing (angular momentum) matrices $s_{\mu\nu}$ and $s^\dagger_{\mu\nu}$ relative to the state vector $\mathcal{X}$; we shall abbreviate this by saying that the states $\mathcal{U}^\pm_\mu(\boldsymbol{k})(\mathcal{X})$ and $\mathcal{U}^{\dagger\pm}_\mu(\boldsymbol{k})(\mathcal{X})$ have spin-mixing matrices $s_{\mu\nu}$ and $s^\dagger_{\mu\nu}$, respectively.

The other additional terms in equations (5.14) are $\pm(x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathbb{1}_4$. They do not mix the components of $\mathcal{U}^\pm_\mu(\boldsymbol{k})(\mathcal{X}_m)$ and $\mathcal{U}^{\dagger\pm}_\mu(\boldsymbol{k})(\mathcal{X}_m)$. These terms may be associated with the orbital angular momentum of the state vectors $\mathcal{U}^\pm_\mu(\boldsymbol{k})(\mathcal{X}_m)$ and $\mathcal{U}^{\dagger\pm}_\mu(\boldsymbol{k})(\mathcal{X}_m)$.

Thus, from (5.12)–(5.14), the following conclusions can be made:

i. The operators $\mathcal{U}^+_\mu(\boldsymbol{k})$ and $\mathcal{U}^{\dagger+}_\mu(\boldsymbol{k})$ (respectively $\mathcal{U}^-_\mu(\boldsymbol{k})$ and $\mathcal{U}^{\dagger-}_\mu(\boldsymbol{k})$) increase (respectively decrease) the state's 4-momentum by the quantity $(\sqrt{m^2c^2+\boldsymbol{k}^2}, \boldsymbol{k})$.

ii. The operators $\mathcal{U}_\mu$, $\mathcal{U}^\pm_\mu$ and $\mathcal{U}^\pm_\mu(\boldsymbol{k})$ (respectively $\mathcal{U}^\dagger_\mu$, $\mathcal{U}^{\dagger\pm}_\mu$ and $\mathcal{U}^{\dagger\pm}_\mu(\boldsymbol{k})$) decrease (respectively increase) the states' charge by q.

iii. The operators $\mathcal{U}^+_\mu(\boldsymbol{k})$ and $\mathcal{U}^{\dagger+}_\mu(\boldsymbol{k})$ (respectively $\mathcal{U}^-_\mu(\boldsymbol{k})$ and $\mathcal{U}^{\dagger-}_\mu(\boldsymbol{k})$) increase (respectively decrease) the state's orbital angular momentum by $(x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}$.

iv. The operators $\mathcal{U}^\pm_\mu(\boldsymbol{k})$ and $\mathcal{U}^{\dagger\pm}_\mu(\boldsymbol{k})$ possess spin-mixing angular momentum matrices $-\mathrm{i}\hbar I_{\mu\nu}$ relative to states with fixed total angular momentum.

In this way, the operators $\mathcal{U}^\pm_\mu(\boldsymbol{k})$ and $\mathcal{U}^{\dagger\pm}_\mu(\boldsymbol{k})$ obtain an interpretation of creation and annihilation operators of particles (quanta) of a vector field, viz.

(a) the operator $\mathcal{U}^+_\mu(\boldsymbol{k})$ (respectively $\mathcal{U}^-_\mu(\boldsymbol{k})$) creates (respectively annihilates) a particle with 4-momentum $(\sqrt{m^2c^2+\boldsymbol{k}^2}, \boldsymbol{k})$, charge $(-q)$ (resp. $(+q)$), orbital angular momentum $(x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}$, and spin-mixing angular momentum matrix $\bigl(-\mathrm{i}\hbar I_{\mu\nu}\bigr)$ and

(b) the operator $\mathcal{U}^{\dagger+}_\mu(\boldsymbol{k})$ (respectively $\mathcal{U}^{\dagger-}_\mu(\boldsymbol{k})$) creates (respectively annihilates) a particle with 4-momentum $(\sqrt{m^2c^2+\boldsymbol{k}^2}, \boldsymbol{k})$, charge $(+q)$ (resp. $(-q)$), orbital angular momentum $(x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}$, and spin-mixing angular momentum matrix $\bigl(-\mathrm{i}\hbar I_{\mu\nu}\bigr)$.

Since $\mathcal{U}^\mu_{s,(\pm)}(k)$ and $\mathcal{U}^{\mu,\dagger}_{s,(\pm)}(k)$ are, by definition, arbitrary linearly independent solutions of (4.15), the operators $\mathcal{U}^\pm_{\mu,s}(\boldsymbol{k})$ and $\mathcal{U}^{\dagger\pm}_{\mu,s}(\boldsymbol{k})$ are linearly independent solutions of the operator equations (see (5.1) and (5.6))

$$k^\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}^\pm_{\mu,s}(\boldsymbol{k}) = 0 \qquad k^\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}^{\dagger\pm}_{\mu,s}(\boldsymbol{k}) = 0, \tag{5.15}$$

which, by virtue of (5.3), imply

$$k^\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}^\pm_{\mu}(\boldsymbol{k}) = 0 \qquad k^\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\mathcal{U}^{\dagger\pm}_{\mu}(\boldsymbol{k}) = 0. \tag{5.16}$$

The fact that the classical vector fields $v^s_\mu(\boldsymbol{k})$ are linearly independent solutions of the same equations (see (4.20)) gives us the possibility to separate the invariant, frame-independent, operator part of the field operators and the frame-dependent their properties by writing (do not sum over s!)

$$\begin{aligned}\mathcal{U}^\pm_{\mu,s}(\boldsymbol{k}) &:= \bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{-1/2}a^\pm_s(\boldsymbol{k})v^s_\mu(\boldsymbol{k})\\ \mathcal{U}^{\dagger\pm}_{\mu,s}(\boldsymbol{k}) &:= \bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{-1/2}a^{\dagger\pm}_s(\boldsymbol{k})v^s_\mu(\boldsymbol{k}),\end{aligned} \tag{5.17}$$

which is equivalent to expend $\mathcal{U}_\mu^\pm$ and $\mathcal{U}_\mu^{\dagger\pm}$ as

$$\begin{aligned}\mathcal{U}_\mu^\pm(\boldsymbol{k}) &:= \bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{-1/2}\sum_{s=1}^{3} a_s^\pm(\boldsymbol{k})v_\mu^s(\boldsymbol{k})\\ \mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k}) &:= \bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{-1/2}\sum_{s=1}^{3} a_s^{\dagger\pm}(\boldsymbol{k})v_\mu^s(\boldsymbol{k}),\end{aligned} \tag{5.18}$$

where $a_s^\pm(\boldsymbol{k}), a_s^{\dagger\pm}(\boldsymbol{k})\colon \mathcal{F}\to\mathcal{F}$ are some operators such that

$$(a_s^\pm(\boldsymbol{k}))^\dagger = a_s^{\dagger\mp}(\boldsymbol{k}) \quad (a_s^{\dagger\pm}(\boldsymbol{k}))^\dagger = a_s^\mp(\boldsymbol{k}), \tag{5.19}$$

due to (5.4). The normalization constant $\bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{-1/2}$ is introduced in (5.17) and (5.18) for future convenience (see Sect. 6). The operators $a_s^+(\boldsymbol{k})$ and $a_s^{\dagger+}(\boldsymbol{k})$ (resp. $a_s^-(\boldsymbol{k})$ and $a_s^{\dagger-}(\boldsymbol{k})$) will be referred as the *creation* (resp. *annihilation*) *operators* (of the field).

The physical meaning of the creation and annihilation operators is similar to the one of $\mathcal{U}_\mu^\pm$ and $\mathcal{U}_\mu^{\dagger\pm}$. To demonstrate this, we insert (5.18) into (5.12)–(5.14) and, using (4.22), we get:

$$\left.\begin{aligned}\mathcal{P}_\mu\bigl(a_s^\pm(\boldsymbol{k})(\mathcal{X}_p)\bigr) &= (p_\mu \pm k_\mu)a_s^\pm(\boldsymbol{k})(\mathcal{X}_p)\\ \mathcal{P}_\mu\bigl(a_s^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_p)\bigr) &= (p_\mu \pm k_\mu)a_s^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_p)\end{aligned}\right\} \quad s = \begin{cases}1,2,3 & \text{for } m\neq 0\\ 1,2 & \text{for } m=0\end{cases} \tag{5.20a}$$

$$\left.\begin{aligned}\mathcal{Q}\bigl(a_s^\pm(\boldsymbol{k})(\mathcal{X}_e)\bigr) &= (e-q)a_s^\pm(\boldsymbol{k})(\mathcal{X}_e)\\ \mathcal{Q}\bigl(a_s^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_e)\bigr) &= (e+q)a_s^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_e)\end{aligned}\right\} \quad s = \begin{cases}1,2,3 & \text{for } m\neq 0\\ 1,2 & \text{for } m=0\end{cases} \tag{5.20b}$$

$$\left.\begin{aligned}\mathcal{M}_{\mu\nu}(x)\bigl(a_s^\pm(\boldsymbol{k})(\mathcal{X}_m)\bigr) &= \{m_{\mu\nu}(x) \pm (x_\mu k_\nu - x_\nu k_\mu)\}a_s^\pm(\boldsymbol{k})(\mathcal{X}_m)\\ &\quad + \mathrm{i}\hbar\sum_{t=1}^{3}\sigma_{\mu\nu}^{st}(\boldsymbol{k})a_t^\pm(\boldsymbol{k})(\mathcal{X}_m)\\ \mathcal{M}_{\mu\nu}(x)\bigl(a_s^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_m)\bigr) &= \{m_{\mu\nu}(x) \pm (x_\mu k_\nu - x_\nu k_\mu)\}a_s^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_m)\\ &\quad + \mathrm{i}\hbar\sum_{t=1}^{3}\sigma_{\mu\nu}^{st}(\boldsymbol{k})a_t^{\dagger\pm}(\boldsymbol{k})(\mathcal{X}_m)\\ \sum_{t=1}^{2}\sigma_{\mu\nu}^{3t}a_t^\pm(\boldsymbol{k}) = 0 \quad \sum_{t=1}^{2}\sigma_{\mu\nu}^{3t}a_t^{\dagger\pm}(\boldsymbol{k}) &= 0 \qquad \text{if } m=0\end{aligned}\right\} \quad s = \begin{cases}1,2,3 & \text{for } m\neq 0\\ 1,2 & \text{for } m=0\end{cases} \tag{5.20c}$$

where

$$\begin{aligned}\sigma_{\mu\nu}^{st}(\boldsymbol{k}) :&= -v^{\lambda,s}(\boldsymbol{k})I_{\lambda\mu\nu}^{\varkappa}v_\varkappa^t(\boldsymbol{k}) = -v^{\lambda,s}(\boldsymbol{k})(\delta_\mu^\varkappa\eta_{\nu\lambda} - \delta_\nu^\varkappa\eta_{\mu\lambda})v_\varkappa^t(\boldsymbol{k})\\ &= v_\mu^s(\boldsymbol{k})v_\nu^t(\boldsymbol{k}) - v_\nu^s(\boldsymbol{k})v_\mu^t(\boldsymbol{k}) = -\sigma_{\nu\mu}^{st}(\boldsymbol{k}) = -\sigma_{\mu\nu}^{ts}(\boldsymbol{k})\end{aligned} \tag{5.21}$$

with $s,t = 1,2,3$.[18]

As a consequence of (5.20), the interpretation of $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ is almost the same as the one of $\mathcal{U}_\mu^\pm(\boldsymbol{k})$ and $\mathcal{U}_\mu^{\dagger\pm}(\boldsymbol{k})$, respectively, with an only change regarding the angular momentum in the massless case. Equations (5.20) do not say anything about the dynamical

[18] Notice, the last equations in (5.20c), valid only in the massless case, impose, generally, 6 conditions on any one of the pairs of operators $a_1^+(\boldsymbol{k})$ and $a_2^+(\boldsymbol{k})$ and $a_1^-(\boldsymbol{k})$ and $a_2^-(\boldsymbol{k})$. However, one should not be worried about that as these conditions originate from the *external* to the Lagrangian formalism equations (2.44) and, consequently, they may not hold in particular theory based on the Lagrangian formalism; see the paragraph containing equations (6.18) below.

characteristics of the states $a_3^+(\mathcal{X}_m)$ and $a_3^{\dagger\,+}(\mathcal{X}_m)$ for a vanishing mass. All this indicates possible problems with the degree of freedom arising from the value $s = 3$ of the polarization variable in the massless case. Indeed, as we shall see below in Sect. 11, this is an 'unphysical' variable; this agrees with the known fact that a massless vector field possesses only two, not three, independent components.

6. The dynamical variables in terms of creation and annihilation operators

The Lagrangian (3.7) (under the Lorenz conditions (3.18)), energy-momentum operator (3.30), current operator (3.31), and orbital angular momentum operator (3.32) are sums of similar ones corresponding to the components $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ of a vector field, considered as independent free scalar fields (see [13]). Besides, the operators $\mathcal{U}_\mu^\pm(\boldsymbol{k})$ and $\mathcal{U}_\mu^{\dagger\,\pm}(\boldsymbol{k})$, defined via (5.5), (5.1) and (5.2), are up to the normalization constant $\bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{1/2}$ equal to the creation/annihilation operators for $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ (considered as independent scalar fields [13]). Consequently, we can automatically write the expressions for the momentum, charge and angular momentum operators in terms of $\mathcal{U}_\mu^\pm(\boldsymbol{k})$ and $\mathcal{U}_\mu^{\dagger\,\pm}(\boldsymbol{k})$ by applying the results obtained in [13] for arbitrary free scalar fields. In this way, we get the following representations for the momentum operator $\mathcal{P}_\mu$, charge operator $\tilde{\mathcal{Q}}$ and orbital angular momentum operator $\tilde{\mathcal{L}}_{\mu\nu}$:[19]

$$
\begin{aligned}
\tilde{\mathcal{P}}_\mu = -\frac{1}{1+\tau(\mathcal{U})}\int \bigl\{k_\mu\bigl(2c(2\pi\hbar)^3k_0\bigr)\bigr\}\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}& \\
\times\bigl\{\mathcal{U}_\lambda^{\dagger\,+}(\boldsymbol{k})\circ\mathcal{U}^{\lambda,-}(\boldsymbol{k})+\mathcal{U}_\lambda^{\dagger\,-}(\boldsymbol{k})\circ\mathcal{U}^{\lambda,+}(\boldsymbol{k})\bigr\}\,\mathrm{d}^3\boldsymbol{k}&
\end{aligned}
\tag{6.1}
$$

$$
\tilde{\mathcal{Q}} = -q\int \mathrm{d}^3\boldsymbol{k}\bigl(2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr)\bigl\{\mathcal{U}_\lambda^{\dagger\,+}(\boldsymbol{k})\circ\mathcal{U}^{\lambda,-}(\boldsymbol{k})-\mathcal{U}_\lambda^{\dagger\,-}(\boldsymbol{k})\circ\mathcal{U}^{\lambda,+}(\boldsymbol{k})\bigr\} \tag{6.2}
$$

$$
\begin{aligned}
\tilde{\mathcal{L}}_{\mu\nu} = x_{0\,\mu}\mathcal{P} - x_{0\,\nu}\mathcal{P} - \frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))}\int \mathrm{d}^3\boldsymbol{k}\bigl(2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr)& \\
\times\Bigl\{\mathcal{U}_\lambda^{\dagger\,+}(\boldsymbol{k})\Bigl(k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}-k_\nu\overleftrightarrow{\frac{\partial}{\partial k^\mu}}\Bigr)\circ\mathcal{U}^{\lambda,-}(\boldsymbol{k})& \\
-\mathcal{U}_\lambda^{\dagger\,-}(\boldsymbol{k})\Bigl(k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}-k_\nu\overleftrightarrow{\frac{\partial}{\partial k^\mu}}\Bigr)\circ\mathcal{U}^{\lambda,+}(\boldsymbol{k})\Bigr\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}},&
\end{aligned}
\tag{6.3}
$$

where

$$
\begin{aligned}
A(\boldsymbol{k})k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}\circ B(\boldsymbol{k}) := -\Bigl(k_\mu\frac{\partial A(\boldsymbol{k})}{\partial k^\nu}\Bigr)\circ B(\boldsymbol{k})+\Bigl(A(\boldsymbol{k})\circ k_\mu\frac{\partial B(\boldsymbol{k})}{\partial k^\nu}\Bigr)& \\
= k_\mu\Bigl(A(\boldsymbol{k})\overleftrightarrow{\frac{\partial}{\partial k^\nu}}\circ B(\boldsymbol{k})\Bigr)&
\end{aligned}
\tag{6.4}
$$

for operators $A(\boldsymbol{k})$ and $B(\boldsymbol{k})$ having C^1 dependence on $\boldsymbol{k}$ (and common domains).[20]

---

[19] The choice of the Lagrangian (3.7) corresponds to the Lagrangian $'\tilde{\mathcal{L}}$ and energy-momentum operator $\tilde{\mathcal{T}}_{\mu\nu}^{(3)}$ in [13]. So, the below-presented operators are consequences of the expressions for $'\mathcal{P}_\mu^{(3)}$, $'\mathcal{Q}_\mu^{(3)}$, and $'\mathcal{L}_{\mu\nu}^{(3)}$, in *loc. cit.*

[20] More generally, if $\omega\colon\{\mathcal{F}\to\mathcal{F}\}\to\{\mathcal{F}\to\mathcal{F}\}$ is a mapping on the operator space over the system's Hilbert space, we put $A\overleftrightarrow{\omega}\circ B := -\omega(A)\circ B + A\circ\omega(B)$ for any $A,B\colon\mathcal{F}\to\mathcal{F}$. Usually [2,16], this notation is used for $\omega=\partial_\mu$.

Now we shall express these operators in terms of the creation and annihilation operators $a_s^{\pm}(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$, introduced in Sect. 5. For the purpose, one should substitute (5.18) into (6.1)–(6.3) and to take into account the normalization conditions (4.22) for the vectors $v_\mu^s(\boldsymbol{k})$, $s = 1, 2, 3$. The result of this procedure reads:

$$\mathcal{P}_\mu = \frac{1}{1+\tau(\mathcal{U})} \sum_{s=1}^{3-\delta_{0m}} \int k_\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{a_s^{\dagger+}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) + a_s^{\dagger-}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{6.5}$$

$$\tilde{\mathcal{Q}} = q \sum_{s=1}^{3-\delta_{0m}} \int \{a_s^{\dagger+}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) - a_s^{\dagger-}(\boldsymbol{k}) \circ a_s^+(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{6.6}$$

$$\begin{aligned}
\tilde{\mathcal{L}}_{\mu\nu} = {} & \frac{1}{1+\tau(\mathcal{U})} \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} (x_{0\,\mu}k_\nu - x_{0\,\nu}k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{a_s^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k}) + a_s^{\dagger-}(\boldsymbol{k})\circ a_s^+(\boldsymbol{k})\} \\
& + \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})} \sum_{s,s'=1}^{3} \int \mathrm{d}^3\boldsymbol{k}\, l_{\mu\nu}^{ss'}(\boldsymbol{k}) \{a_s^{\dagger+}(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) - a_s^{\dagger-}(\boldsymbol{k}) \circ a_{s'}^+(\boldsymbol{k})\} \\
& + \frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))} \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a_s^{\dagger+}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^-(\boldsymbol{k}) \\
& \qquad - a_s^{\dagger-}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^+(\boldsymbol{k}) \Big\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} ,
\end{aligned} \tag{6.7}$$

where

$$\begin{aligned}
l_{\mu\nu}^{ss'}(\boldsymbol{k}) : &= \frac{1}{2} v_\lambda^s(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) v^{\lambda,s'}(\boldsymbol{k}) = -l_{\nu\mu}^{ss'}(\boldsymbol{k}) = -l_{\mu\nu}^{s's}(\boldsymbol{k}) \\
&= -\Big(k_\mu \frac{\partial v_\lambda^s(\boldsymbol{k})}{\partial k^\nu} - k_\nu \frac{\partial v_\lambda^s(\boldsymbol{k})}{\partial k^\mu} \Big) v^{\lambda,s'}(\boldsymbol{k}) \\
&= +v_\lambda^s(\boldsymbol{k}) \Big(k_\mu \frac{\partial v^{\lambda,s'}(\boldsymbol{k})}{\partial k^\nu} - k_\nu \frac{\partial v^{\lambda,s'}(\boldsymbol{k})}{\partial k^\mu} \Big).
\end{aligned} \tag{6.8}$$

with the restriction $k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2}$ done after the differentiation (so that the derivatives with respect to k_0 vanish). The last two equalities in (6.8) are consequences of (see (4.22))

$$\frac{\partial v_\mu^s(\boldsymbol{k})}{\partial k^\lambda} v^{\mu,s'}(\boldsymbol{k}) + v_\mu^s(\boldsymbol{k}) \frac{\partial v^{\mu,s'}(\boldsymbol{k})}{\partial k^\lambda} = 0, \tag{6.9}$$

so that

$$v_\lambda^s(\boldsymbol{k}) k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} v^{\lambda,s'}(\boldsymbol{k}) = -2k_\mu \frac{\partial v_\lambda^s(\boldsymbol{k})}{\partial k^\nu} v^{\lambda,s'}(\boldsymbol{k}) = 2k_\mu v_\lambda^s(\boldsymbol{k}) \frac{\partial v^{\lambda,s'}(\boldsymbol{k})}{\partial k^\nu}. \tag{6.10}$$

Since, $v_\mu^s(\boldsymbol{k})$ are real (see (4.19)–(4.25)), the definition (6.8) implies

$$\big(l_{\mu\nu}^{ss'}(\boldsymbol{k})\big)^* = l_{\mu\nu}^{ss'}(\boldsymbol{k}) = -l_{\mu\nu}^{s's}(\boldsymbol{k}), \tag{6.11}$$

where the asterisk $*$ denotes complex conjugation. So, $l_{\mu\nu}^{ss'}(\boldsymbol{k})$ are real and, by virtue of (5.19), the sums of the first/second terms in the last integrand in (6.7) are Hermitian.

A peculiarity of (6.5)–(6.7) is the presence in them of the Kronecker symbol δ_{0m}, which equals to zero in the massive case, $m \neq 0$, and to one in the massless case, $m = 0$. Thus, in the massless case, the modes with polarization $s = 3$ do *not* contribute to the momentum

and charge operators, but they *do* contribute to the orbital angular momentum operator only via the numbers (6.8) in the second term in (6.7). Notice, in this way the arbitrariness in the definition (4.20)–(4.21) of the vectors $v_\mu^s(\boldsymbol{k})$ enters in the orbital angular momentum operator. This is more or less an expected conclusion as the last operator is generally a frame-dependent object.

Let us turn now our attention to the spin angular momentum operator (2.25) with density operator $\mathcal{S}_{\mu\nu}^\lambda$ given by (3.33). Substituting (5.6)–(5.7) into (3.33), we get the following representation of the spin angular momentum density operator:

$$\begin{aligned}\mathcal{S}_{\mu\nu}^\lambda = -\frac{\mathrm{i}\hbar c^2}{1+\tau(\mathcal{U})}\int \mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'\{k^\lambda|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\big(-\mathcal{U}_\mu^{\dagger\,+}(\boldsymbol{k})+\mathcal{U}_\mu^{\dagger\,-}(\boldsymbol{k})\big)\circ\big(\mathcal{U}_\nu^+(\boldsymbol{k}')+\mathcal{U}_\nu^-(\boldsymbol{k}')\big)\\ - k^\lambda|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\big(-\mathcal{U}_\nu^{\dagger\,+}(\boldsymbol{k})+\mathcal{U}_\nu^{\dagger\,-}(\boldsymbol{k})\big)\circ\big(\mathcal{U}_\mu^+(\boldsymbol{k}')+\mathcal{U}_\mu^-(\boldsymbol{k}')\big)\\ - k'^\lambda|_{k_0'=\sqrt{m^2c^2+\boldsymbol{k}'^2}}\big(\mathcal{U}_\mu^{\dagger\,+}(\boldsymbol{k})+\mathcal{U}_\mu^{\dagger\,-}(\boldsymbol{k})\big)\circ\big(-\mathcal{U}_\nu^+(\boldsymbol{k}')+\mathcal{U}_\nu^-(\boldsymbol{k}')\big)\\ + k^\lambda|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\big(\mathcal{U}_\nu^{\dagger\,+}(\boldsymbol{k})+\mathcal{U}_\nu^{\dagger\,-}(\boldsymbol{k})\big)\circ\big(-\mathcal{U}_\mu^+(\boldsymbol{k}')+\mathcal{U}_\mu^-(\boldsymbol{k}')\big).\quad(6.12)\end{aligned}$$

We shall calculate the spin angular momentum operator in Heisenberg picture by inserting (6.12), with $\lambda=0$, into (2.25). Then one should 'move' the operators $\mathcal{U}_\mu^\pm$ and $\mathcal{U}_\mu^{\dagger\,\pm}$ to the right of $\mathcal{U}(x,x_0)$ according to the relation

$$\varphi^\varepsilon(\boldsymbol{k})\circ\varphi^{\varepsilon'}(\boldsymbol{k}')\circ\mathcal{U}(x,x_0) = \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}(x^\mu-x_0^\mu)(\varepsilon k_\mu+\varepsilon' k'_\mu)}\,\mathcal{U}(x,x_0)\varphi^\varepsilon(\boldsymbol{k})\circ\varphi^{\varepsilon'}(\boldsymbol{k}')\qquad(6.13)$$

where $\varepsilon,\varepsilon' = +,-$, $k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}$, $k_0'=\sqrt{m^2c^2+(\boldsymbol{k}')^2}$, $\varphi^\varepsilon(\boldsymbol{k})=\mathcal{U}_\mu^\varepsilon(\boldsymbol{k}),\mathcal{U}_\mu^\varepsilon(\boldsymbol{k})$, and $\mathcal{U}(x,x_0)$ being the operator (2.2) by means of which the transition from Heisenberg to momentum picture is performed. This relation is valid for any $\varphi^\varepsilon(\boldsymbol{k})$ such that $[\varphi^\varepsilon(\boldsymbol{k}),\mathcal{P}_\nu]_\_ = -k_\nu\varphi^\varepsilon(\boldsymbol{k})$ – see (5.8) and [13, eq. (6.4)]. At last, performing the integration over $\boldsymbol{x}$, which results in the terms $(2\pi\hbar)^3\delta^3(\boldsymbol{k}\pm\boldsymbol{k}')$, and the trivial integration over $\boldsymbol{k}'$ by means of the δ-functions $\delta^3(\boldsymbol{k}\pm\boldsymbol{k}')$, we find:

$$\begin{aligned}\tilde{\mathcal{S}}_{\mu\nu} = -\frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\int \mathrm{d}^3\boldsymbol{k}\big(2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\big)\{-\mathcal{U}_\mu^{\dagger\,+}(\boldsymbol{k})\circ\mathcal{U}_\nu^-(\boldsymbol{k})+\mathcal{U}_\mu^{\dagger\,-}(\boldsymbol{k})\circ\mathcal{U}_\nu^+(\boldsymbol{k})\\ +\mathcal{U}_\nu^{\dagger\,+}(\boldsymbol{k})\circ\mathcal{U}_\mu^-(\boldsymbol{k})-\mathcal{U}_\nu^{\dagger\,-}(\boldsymbol{k})\circ\mathcal{U}_\mu^+(\boldsymbol{k})\}.\quad(6.14)\end{aligned}$$

To express $\tilde{\mathcal{S}}_{\mu\nu}$ via the creation and annihilation operators, we substitute (5.18) into (6.14) and get

$$\tilde{\mathcal{S}}_{\mu\nu} = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\sum_{s,s'=1}^{3}\int \mathrm{d}^3\boldsymbol{k}\,\sigma_{\mu\nu}^{ss'}(\boldsymbol{k})\{a_s^{\dagger\,+}(\boldsymbol{k})\circ a_{s'}^-(\boldsymbol{k}) - a_s^{\dagger\,-}(\boldsymbol{k})\circ a_{s'}^+(\boldsymbol{k})\},\qquad(6.15)$$

where the functions

$$\sigma_{\mu\nu}^{st}(\boldsymbol{k}) := v_\mu^s(\boldsymbol{k})v_\nu^t(\boldsymbol{k}) - v_\nu^s(\boldsymbol{k})v_\mu^t(\boldsymbol{k}) = -\sigma_{\nu\mu}^{st}(\boldsymbol{k}) = -\sigma_{\mu\nu}^{ts}(\boldsymbol{k}) = +\sigma_{\nu\mu}^{ts}(\boldsymbol{k})\qquad(6.16)$$

were introduced earlier by (5.21).

From (6.15), we observe that $\tilde{\mathcal{S}}_{\mu\nu}$, generally, depends on the mode with polarization index $s=3$ even in the massless case. Evidently, a necessary and sufficient condition for the independence of the spin angular momentum form this mode is

$$\begin{aligned}\sum_{s=1,2}\int \mathrm{d}^3\boldsymbol{k}\{\big(\sigma_{\mu\nu}^{s3}(\boldsymbol{k})a_s^{\dagger\,+}(\boldsymbol{k})\big)\circ a_3^-(\boldsymbol{k}) - \big(\sigma_{\mu\nu}^{s3}(\boldsymbol{k})a_s^{\dagger\,-}(\boldsymbol{k})\big)\circ a_3^+(\boldsymbol{k})\\ + a_3^{\dagger\,+}(\boldsymbol{k})\circ\big(\sigma_{\mu\nu}^{3s}(\boldsymbol{k})a_s^-(\boldsymbol{k})\big) - a_3^{\dagger\,-}(\boldsymbol{k})\circ\big(\sigma_{\mu\nu}^{3s}(\boldsymbol{k})a_s^+(\boldsymbol{k})\big)\} = 0.\quad(6.17)\end{aligned}$$

In particular, (6.17) is fulfilled in the massless case if

$$\sum_{s=1}^{2} \sigma_{\mu\nu}^{3s} a_s^{\pm}(\boldsymbol{k}) = 0 \quad \sum_{s=1}^{2} \sigma_{\mu\nu}^{3s} a_s^{\dagger\,\pm}(\boldsymbol{k}) = 0 \qquad \text{for } m = 0, \tag{6.18}$$

which are exactly the conditions appearing in (5.20c).[21] Thus, a strange situation arises: the massless modes with $s = 3$ do *not* contribute to the momentum and charge operators, but they *do* contribute to the spin and orbital angular momentum operators unless additional conditions, like (6.18), are valid. However, one can prove that (6.18) is either equivalent to $a_s^{\pm}(\boldsymbol{k}) = a_s^{\dagger\,\pm}(\boldsymbol{k}) = 0$, if $s = 1, 2$ and $(\boldsymbol{k}, m) \neq (\boldsymbol{0}, 0)$, or it is identically valid, if $(\boldsymbol{k}, m) = (\boldsymbol{0}, 0)$.[22] In view of (6.5)–(6.8) and (6.15), this means that (6.18) may be valid only for free vector fields with vanishing momentum, charge and spin and angular momentum operators, which fields are completely unphysical as they cannot lead to any physically observable/predictable results. Thus, only for such unphysical massless free vector fields the Heisenberg relations (3.35) (for the Lagrangian (3.7)) may be valid. For these reasons, the relations (3.35) and (6.18) will not be considered further in the present work.

However, the above conclusions do not exclude the validity of the more general than (6.18) equation (6.17). We shall return on this problem in Sect. 11.

To get a concrete idea of the spin angular momentum operator (6.15), we shall calculate the quantities (6.16) for particular choices of the vectors $e_\mu^1(\boldsymbol{k})$ and $e_\mu^2(\boldsymbol{k})$ in (4.23)–(4.24).

To begin with, we notice that, as a result of the antisymmetry of $\sigma_{\mu\nu}^{st}(\boldsymbol{k})$ in the superscripts and subscripts, only $\frac{3(3-1)}{2} \times \frac{4(4-1)}{2} = 18$ of all of the $3^2 \times 4^2 = 144$ of these quantities are independent. As such we shall choose the ones with $(s, t) = (1, 2), (2, 3), (3, 1)$ and $(\mu, \nu) = (0, 1), (0, 2), (0, 3), (1, 2), (2, 3), (3, 1)$.

For $\boldsymbol{k} \neq \boldsymbol{0}$, we choose a frame such that

$$\boldsymbol{k} = (0, 0, 0, k^3)\big|_{k^3 = \sqrt{\boldsymbol{k}^2} \geq 0} \quad e_a^1(\boldsymbol{k}) = -\delta_a^1 \quad e_a^2(\boldsymbol{k}) = -\delta_a^2. \tag{6.19}$$

Let $\Lambda := \frac{k^3}{\sqrt{\boldsymbol{k}^2}} \Big(\frac{m^2c^2 + \boldsymbol{k}^2}{m^2c^2 + \boldsymbol{k}^2\delta_{0m}} \Big)^{1/2} = v^{3,3}(\boldsymbol{k}) = -v_3^3(\boldsymbol{k})$. The results of a straightforward calculation, by means of (6.16), of the chosen independent quantities $\sigma_{\mu\nu}^{st}(\boldsymbol{k})$ are presented in table 6.1. Similar results for $\boldsymbol{k} = \boldsymbol{0}$ in a frame in which

$$v_0^s(\boldsymbol{0}) = 0 \quad e_a^1(\boldsymbol{0}) = -\delta_a^1 \quad e_a^2(\boldsymbol{0}) = -\delta_a^2 \quad v_a^3(\boldsymbol{0}) = -\delta_a^3(1 - \delta_{0m}) \tag{6.20}$$

are given in table 6.2.

Consider now the so-called spin vector(s). Since $\tilde{\mathcal{S}}_{\mu\nu}$ is antisymmetric in μ and ν, $\tilde{\mathcal{S}}_{\mu\nu} = -\tilde{\mathcal{S}}_{\nu\mu}$, the spin angular momentum operator has 6 independent components, from which can be formed two 3-dimensional vectors, viz.

$$\tilde{\mathcal{R}}_a := \tilde{\mathcal{S}}_{0a} \qquad \tilde{\mathcal{S}}^a := \frac{1}{2}\varepsilon^{abc}\,\tilde{\mathcal{S}}^{bc} = \sum_{b,c=1}^{3} \frac{1}{2}\varepsilon^{abc}\,\tilde{\mathcal{S}}^{bc} \tag{6.21}$$

---

[21] Recall, the equation (6.18) was derived in Sect. 5 on the base of the relations (3.35), which are external to the Lagrangian formalism.

[22] Let $\boldsymbol{k} \neq \boldsymbol{0}$. Substituting (6.16), with $t = 3$, into the first equation in (6.18) and, then, using (4.23), we find

$$(k_a e_b^1(\boldsymbol{k}) - k_b e_a^1(\boldsymbol{k})) a_1^{\pm}(\boldsymbol{k}) + (k_a e_b^2(\boldsymbol{k}) - k_b e_a^2(\boldsymbol{k})) a_2^{\pm}(\boldsymbol{k}) = 0 \quad a, b = 1, 2, 3. \tag{*}$$

Multiplying this equality with $e_a^1(\boldsymbol{k})$ or $e_a^2(\boldsymbol{k})$ and summing over $a = 1, 2, 3$, we, in view of (4.24), get $k_b a_1^{\pm}(\boldsymbol{k}) = k_b a_2^{\pm}(\boldsymbol{k}) = 0$ for any $b = 1, 2, 3$. Therefore $a_1^{\pm}(\boldsymbol{k}) = a_2^{\pm}(\boldsymbol{k}) = 0$, as we supposed $\boldsymbol{k} \neq 0$. If $\boldsymbol{k} = 0$, repeating the above method by using (4.25) for (4.23), we get

$$(1 - \delta_{0m})\{(\delta_a^3 e_b^1(\boldsymbol{0}) - \delta_b^3 e_a^1(\boldsymbol{0})) a_1^{\pm}(\boldsymbol{0}) + (\delta_a^3 e_b^2(\boldsymbol{0}) - \delta_b^3 e_a^2(\boldsymbol{0})) a_2^{\pm}(\boldsymbol{0})\} = 0 \tag{**}$$

instead of (*). This equation is identically valid for $m = 0$, but for $m \neq 0$ it, by virtue of (4.24), implies $a_1^{\pm}(\boldsymbol{k}) = a_2^{\pm}(\boldsymbol{k}) = 0$. The assertion for the operators $a_1^{\dagger\,\pm}(\boldsymbol{k})$ and $a_2^{\dagger\,\pm}(\boldsymbol{k})$ can be proved similarly; alternatively, it follow from the just proved results and (5.19).

Table 6.1: The quantities (6.16) for $\boldsymbol{k} \neq \boldsymbol{0}$ in the basis (6.19).

$(s,t) \downarrow \xrightarrow{(\mu,\nu)}$	(0,1)	(0,2)	(0,3)	(1,2)	(2,3)	(3,1)
(1,2)	0	0	0	1	0	0
(2,3)	0	$\frac{k^3}{\sqrt{m^2c^2+\boldsymbol{k}^2}}\Lambda$	0	0	Λ	0
(3,1)	$-\frac{k^3}{\sqrt{m^2c^2+\boldsymbol{k}^2}}\Lambda$	0	0	0	0	Λ

Table 6.2: The quantities (6.16) for $\boldsymbol{k} = \boldsymbol{0}$ in the basis (6.20).

$(s,t) \downarrow \xrightarrow{(\mu,\nu)}$	(0,1)	(0,2)	(0,3)	(1,2)	(2,3)	(3,1)
(1,2)	0	0	0	1	0	0
(2,3)	0	0	0	0	$1-\delta_{0m}$	0
(3,1)	0	0	0	0	0	$1-\delta_{0m}$

where ε^{abc} is the 3-dimensional Levi-Civita's symbol (which equals to +1 (resp. -1) if (a,b,c) is an even (resp. odd) permutation of $(1,2,3)$ and to zero otherwise). Defining the cross (vector) product of 3-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in Cartesian coordinates by $(\boldsymbol{A}\times\boldsymbol{B})^a := \varepsilon^{abc}A_bB_c$, where $A_b = -A^b$ are the covariant Cartesian components of $\boldsymbol{A} = (A^1,A^2,A^3)$, from (6.14) we find

$$\tilde{\boldsymbol{\mathcal{R}}} = -\frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\int \mathrm{d}^3\boldsymbol{k}\big(2c(2\pi\hbar\sqrt{m^2c^2+\boldsymbol{k}^2})^3\big)\big\{-\mathcal{U}_0^{\dagger\,+}(\boldsymbol{k})\circ\boldsymbol{\mathcal{U}}^-(\boldsymbol{k})+\mathcal{U}_0^{\dagger\,-}(\boldsymbol{k})\circ\boldsymbol{\mathcal{U}}^+(\boldsymbol{k}) + \boldsymbol{\mathcal{U}}^{\dagger\,+}(\boldsymbol{k})\circ\mathcal{U}_0^-(\boldsymbol{k})-\boldsymbol{\mathcal{U}}^{\dagger\,-}(\boldsymbol{k})\circ\mathcal{U}_0^+(\boldsymbol{k})\big\} \tag{6.22}$$

$$\tilde{\boldsymbol{\mathcal{S}}} = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\int \mathrm{d}^3\boldsymbol{k}\big(2c(2\pi\hbar\sqrt{m^2c^2+\boldsymbol{k}^2})^3\big)\big\{\boldsymbol{\mathcal{U}}^{\dagger\,+}(\boldsymbol{k})\overset{\circ}{\times}\boldsymbol{\mathcal{U}}^-(\boldsymbol{k})-\boldsymbol{\mathcal{U}}^{\dagger\,-}(\boldsymbol{k})\overset{\circ}{\times}\boldsymbol{\mathcal{U}}^+(\boldsymbol{k})\big\}, \tag{6.23}$$

where $\boldsymbol{\mathcal{U}}^{\pm}(\boldsymbol{k}) := (\mathcal{U}^{1,\pm}(\boldsymbol{k}),\mathcal{U}^{2,\pm}(\boldsymbol{k}),\mathcal{U}^{3,\pm}(\boldsymbol{k})) := -(\mathcal{U}_1^{\pm}(\boldsymbol{k}),\mathcal{U}_2^{\pm}(\boldsymbol{k}),\mathcal{U}_3^{\pm}(\boldsymbol{k}))$, $\boldsymbol{\mathcal{U}}^{\dagger\,\pm}(\boldsymbol{k}) := -(\mathcal{U}_1^{\dagger\,\pm}(\boldsymbol{k}),\mathcal{U}_2^{\dagger\,\pm}(\boldsymbol{k}),\mathcal{U}_3^{\dagger\,\pm}(\boldsymbol{k}))$, and $\overset{\circ}{\times}$ means a cross product combined with operator composition, e.g. $(\boldsymbol{A}\overset{\circ}{\times}\boldsymbol{B})^1 = A_2\circ B_3 - A_3\circ B_2$ for operator-valued vectors $\boldsymbol{A}$ and $\boldsymbol{B}$.

To write the spin vectors $\tilde{\boldsymbol{\mathcal{R}}}$ and $\tilde{\boldsymbol{\mathcal{S}}}$ in terms of creation and annihilation operators, we notice that the vectors $\boldsymbol{e}^s(\boldsymbol{k}) := -(e_1^s(\boldsymbol{k}),e_2^s(\boldsymbol{k}),e_3^s(\boldsymbol{k}))$, $s=1,2,3$, with $\boldsymbol{e}^3(\boldsymbol{k}) := \boldsymbol{e}^1(\boldsymbol{k})\times\boldsymbol{e}^2(\boldsymbol{k})$ form an orthonormal basis of the $\mathbb{R}^3$ $\boldsymbol{k}$-space, such that (see (4.23)–(4.25) and do not sum over s)

$$\begin{gathered}\boldsymbol{v}^s(\boldsymbol{k}) := -(v_1^s(\boldsymbol{k}),v_2^s(\boldsymbol{k}),v_3^s(\boldsymbol{k})) = \omega^s(\boldsymbol{k})\boldsymbol{e}^s(\boldsymbol{k})\\ \omega^1(\boldsymbol{k}) = \omega^2(\boldsymbol{k}) := 1 \quad \omega^3(\boldsymbol{k})\big|_{\boldsymbol{k}\neq\boldsymbol{0}} := \frac{m^2c^2+\boldsymbol{k}^2}{m^2c^2+\boldsymbol{k}^2\delta_{0m}} \quad \omega^3(\boldsymbol{0}) := \begin{cases}1 & \text{for } m\neq 0\\ 0 & \text{for } m=0\end{cases}.\end{gathered} \tag{6.24}$$

Then, we have

$$\begin{aligned}\frac{1}{2}\varepsilon^{abc}\sigma_{bc}^{st}(\boldsymbol{k}) &= \varepsilon^{abc}v_b^s(\boldsymbol{k})v_c^t(\boldsymbol{k}) = \omega^s(\boldsymbol{k})\omega^t(\boldsymbol{k})\varepsilon^{abc}e_b^s(\boldsymbol{k})e_c^t(\boldsymbol{k}) = \omega^s(\boldsymbol{k})\omega^t(\boldsymbol{k})\big(\boldsymbol{e}^s(\boldsymbol{k})\times\boldsymbol{e}^t(\boldsymbol{k})\big)^a\\ \sigma_{0a}^{st}(\boldsymbol{k})\big|_{\boldsymbol{k}\neq\boldsymbol{0}} &= v_0^s(\boldsymbol{k})v_a^t(\boldsymbol{k}) - v_a^s(\boldsymbol{k})v_0^t(\boldsymbol{k}) = -\frac{k^b}{\sqrt{m^2c^2+\boldsymbol{k}^2}}\sigma_{ba}^{st}(\boldsymbol{k})\\ &= -\frac{k^b}{\sqrt{m^2c^2+\boldsymbol{k}^2}}\frac{1}{2}\varepsilon_{bac}\varepsilon^{cdf}\sigma_{df}^{st}(\boldsymbol{k}) = +\frac{k^b}{\sqrt{m^2c^2+\boldsymbol{k}^2}}\varepsilon_{abc}(\boldsymbol{v}^s(\boldsymbol{k})\times\boldsymbol{v}^t(\boldsymbol{k}))^c\\ \sigma_{0a}^{st}(\boldsymbol{0}) &= 0,\end{aligned}$$

where (4.23), (4.25), and the equality $\varepsilon_{abc}\varepsilon^{efc} = \delta^e_a\delta^f_b - \delta^e_b\delta^f_a$ were applied. Therefore, from (6.21) and (6.15), we get

$$\tilde{\mathcal{R}}_a = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\varepsilon_{abc}\int \mathrm{d}^3\boldsymbol{k}\, r^b(\boldsymbol{k})\{\boldsymbol{a}^{\dagger\,+}(\boldsymbol{k}) \overset{\circ}{\times} \boldsymbol{a}^-(\boldsymbol{k}) - \boldsymbol{a}^{\dagger\,-}(\boldsymbol{k}) \overset{\circ}{\times} \boldsymbol{a}^+(\boldsymbol{k})\}^c \tag{6.25}$$

$$\tilde{\boldsymbol{\mathcal{S}}} = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\int \mathrm{d}^3\boldsymbol{k}\{\boldsymbol{a}^{\dagger\,+}(\boldsymbol{k}) \overset{\circ}{\times} \boldsymbol{a}^-(\boldsymbol{k}) - \boldsymbol{a}^{\dagger\,-}(\boldsymbol{k}) \overset{\circ}{\times} \boldsymbol{a}^+(\boldsymbol{k})\}, \tag{6.26}$$

where the operator-valued vectors

$$\boldsymbol{a}^{\pm}(\boldsymbol{k}) := \sum_{s=1}^{3} \boldsymbol{v}^s(\boldsymbol{k}) a_s^{\pm}(\boldsymbol{k}) \quad \boldsymbol{a}^{\dagger\,\pm}(\boldsymbol{k}) := \sum_{s=1}^{3} \boldsymbol{v}^s(\boldsymbol{k}) a_s^{\dagger\,\pm}(\boldsymbol{k}) \tag{6.27}$$

were introduced, the index c in $\{\cdots\}^c$ means the c^{th} component of $\{\cdots\}$, and the function

$$r^b(\boldsymbol{k}) := \begin{cases} \frac{k^b}{\sqrt{m^2c^2+\boldsymbol{k}^2}} & \text{for } (\boldsymbol{k}, m) \neq (\boldsymbol{0}, 0) \\ 0 & \text{for } (\boldsymbol{k}, m) = (\boldsymbol{0}, 0) \end{cases} \tag{6.28}$$

takes care of the above-obtained expressions for $\sigma^{st}_{0a}(\boldsymbol{k})$.

In connection with the particle interpretation of the creation and annihilation operators, the component $\tilde{\mathcal{S}}^3$ is of particular interest as, for $\boldsymbol{k} \neq \boldsymbol{0}$, its integrand describes the spin projection on the direction of the 4-momentum $\boldsymbol{k}$. From (6.26), we obtain

$$\begin{aligned} \tilde{\mathcal{S}}^3 = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\int \mathrm{d}^3\boldsymbol{k}\{ & a_1^{\dagger\,+}(\boldsymbol{k}) \circ a_2^-(\boldsymbol{k}) - a_2^{\dagger\,+}(\boldsymbol{k}) \circ a_1^-(\boldsymbol{k}) \\ & - a_1^{\dagger\,-}(\boldsymbol{k}) \circ a_2^+(\boldsymbol{k}) + a_2^{\dagger\,-}(\boldsymbol{k}) \circ a_1^+(\boldsymbol{k})\}, \end{aligned} \tag{6.29}$$

Since in (6.29) enter only 'mixed' products, like $a_1^{\dagger\,\pm}(\boldsymbol{k}) \circ a_2^{\mp}(\boldsymbol{k})$, the states/(anti)particles created/annihilated by $a_s^{\pm}(\boldsymbol{k})$ and $a_s^{\dagger\,\pm}(\boldsymbol{k})$ for $s = 1,2$ do not have definite projection on $\boldsymbol{k}$ (for $\boldsymbol{k} \neq \boldsymbol{0}$). This situation can be improved by introducing operators $b_s^{\pm}(\boldsymbol{k})$ and $b_s^{\dagger\,\pm}(\boldsymbol{k})$ such that [1, eq. (4.28)]

$$\begin{gathered} a_1^{\pm}(\boldsymbol{k}) = \frac{b_1^{\pm}(\boldsymbol{k}) + b_2^{\pm}(\boldsymbol{k})}{\sqrt{2}} \quad a_2^{\pm}(\boldsymbol{k}) = -\mathrm{i}\,\frac{b_1^{\pm}(\boldsymbol{k}) - b_2^{\pm}(\boldsymbol{k})}{\sqrt{2}} \quad a_3^{\pm}(\boldsymbol{k}) = b_3^{\pm}(\boldsymbol{k}) \\ a_1^{\dagger\,\pm}(\boldsymbol{k}) = \frac{b_1^{\dagger\,\pm}(\boldsymbol{k}) + b_2^{\dagger\,\pm}(\boldsymbol{k})}{\sqrt{2}} \quad a_2^{\dagger\,\pm}(\boldsymbol{k}) = +\mathrm{i}\,\frac{b_1^{\dagger\,\pm}(\boldsymbol{k}) - b_2^{\dagger\,\pm}(\boldsymbol{k})}{\sqrt{2}} \quad a_3^{\dagger\,\pm}(\boldsymbol{k}) = b_3^{\dagger\,\pm}(\boldsymbol{k}). \end{gathered} \tag{6.30}$$

In terms of the operators $b_s^{\pm}(\boldsymbol{k})$ and $b_s^{\dagger\,\pm}(\boldsymbol{k})$, the momentum (6.5), charge (6.6) and third spin vector projection (6.29) operators take respectively the forms:

$$\mathcal{P}_\mu = \frac{1}{1+\tau(\mathcal{U})}\sum_{s=1}^{3-\delta_{0m}}\int k_\mu|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\{b_s^{\dagger\,+}(\boldsymbol{k}) \circ b_s^-(\boldsymbol{k}) + b_s^{\dagger\,-}(\boldsymbol{k}) \circ b_s^+(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{6.31}$$

$$\tilde{\mathcal{Q}} = q\sum_{s=1}^{3-\delta_{0m}}\int \{b_s^{\dagger\,+}(\boldsymbol{k}) \circ b_s^-(\boldsymbol{k}) - b_s^{\dagger\,-}(\boldsymbol{k}) \circ b_s^+(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{6.32}$$

$$\tilde{\mathcal{S}}^3 = \frac{\hbar}{1+\tau(\mathcal{U})}\sum_{s=1}^{2}\int (-1)^{s+1}\{b_s^{\dagger\,+}(\boldsymbol{k}) \circ b_s^-(\boldsymbol{k}) - b_s^{\dagger\,-}(\boldsymbol{k}) \circ b_s^+(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k}. \tag{6.33}$$

From these formulae is clear that the states (particles) created/annihilated by $b_s^{\pm}(\boldsymbol{k})$ and $b_s^{\dagger\,\pm}(\boldsymbol{k})$ have 4-momentum $(\sqrt{m^2c^2+\boldsymbol{k}^2}, \boldsymbol{k})$, charge $\pm q$, and spin projection on the direction of movement equal to $\pm\hbar \times (1+\tau(\mathcal{U}))^{-1}$ for $s = 1,2$ or equal to zero if $s = 3$.[23]

[23] One can get rid of the factor $(1+\tau(\mathcal{U}))^{-1}$ by rescaling the operators $b_s^{\pm}(\boldsymbol{k})$ and $b_s^{\dagger\,\pm}(\boldsymbol{k})$ by the factor $(1+\tau(\mathcal{U}))^{1/2}$.

We would like now to make a comparison with the expressions for the dynamical variables in terms of the creation/annihilation operators $\tilde{a}_s^{\pm}(\boldsymbol{k})$ and $\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})$ in (the momentum representation of) Heisenberg picture of motion [1–3,16]. As a consequence of (4.14), the analogues of the creation/annihilation operators, defined in terms of the vector field frequency operators via (5.1) and (5.2), are

$$
\begin{aligned}
\tilde{\mathcal{U}}_{\mu,s}^{\pm}(\boldsymbol{k}) = \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x_0^{\mu}k_{\mu}}\,\mathcal{U}_{\mu,s}^{\pm}(\boldsymbol{k}) \quad \tilde{\mathcal{U}}_{\mu,s}^{\dagger\,\pm}(\boldsymbol{k}) = \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x_0^{\mu}k_{\mu}}\,\mathcal{U}_{\mu,s}^{\dagger\,\pm}(\boldsymbol{k}) \quad \bigl(k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr)\\
\tilde{\mathcal{U}}_{\mu}^{\pm}(\boldsymbol{k}) = \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x_0^{\mu}k_{\mu}}\,\mathcal{U}_{\mu}^{\pm}(\boldsymbol{k}) \quad \tilde{\mathcal{U}}_{\mu}^{\dagger\,\pm}(\boldsymbol{k}) = \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x_0^{\mu}k_{\mu}}\,\mathcal{U}_{\mu}^{\dagger\,\pm}(\boldsymbol{k}) \quad \bigl(k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr)
\end{aligned}
\tag{6.34}
$$

in Heisenberg picture. Therefore, defining (cf. (5.17))

$$
\begin{aligned}
\tilde{\mathcal{U}}_{\mu,s}^{\pm}(\boldsymbol{k}) &=: \bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{-1/2}\tilde{a}_s^{\pm}(\boldsymbol{k})v_{\mu}^{s}(\boldsymbol{k})\\
\tilde{\mathcal{U}}_{\mu,s}^{\dagger\,\pm}(\boldsymbol{k}) &=: \bigl\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr\}^{-1/2}\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})v_{\mu}^{s}(\boldsymbol{k}),
\end{aligned}
\tag{6.35}
$$

we get the creation/annihilation operators in Heisenberg picture as

$$
\tilde{a}_s^{\pm}(\boldsymbol{k}) = \mathrm{e}^{\pm\frac{1}{\mathrm{i}\hbar}x_0^{\mu}k_{\mu}}a_s^{\pm}(\boldsymbol{k}) \quad \tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k}) = e^{\pm\frac{1}{\mathrm{i}\hbar}x_0^{\mu}k_{\mu}}a_s^{\dagger\,\pm}(\boldsymbol{k}) \quad \bigl(k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}\bigr).
\tag{6.36}
$$

Evidently, these operators satisfy the equations

$$
\bigl(\tilde{a}_s^{\pm}(\boldsymbol{k})\bigr)^{\dagger} = \tilde{a}_s^{\dagger\,\mp}(\boldsymbol{k}) \quad \bigl(\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})\bigr)^{\dagger} = \tilde{a}_s^{\mp}(\boldsymbol{k}),
\tag{6.37}
$$

due to (5.19), and have all other properties of their momentum picture counterparts described in Sect. 5.

The connection (2.4) is not applicable to the creation/annihilation operators, as well as to operators in momentum representation (of momentum picture), i.e. to ones depending on the momentum variable $\boldsymbol{k}$. In particular, the reader may verify, by using the results of sections 4 and 5, the formulae

$$
\left.
\begin{aligned}
a_s^{\pm}(\boldsymbol{k}) &= \mathrm{e}^{\mp\frac{1}{\mathrm{i}\hbar}x^{\mu}k_{\mu}}\,\mathcal{U}(x,x_0)\circ\tilde{a}_s^{\pm}(k)\circ\mathcal{U}^{-1}(x,x_0)\\
a_s^{\dagger\,\pm}(\boldsymbol{k}) &= \mathrm{e}^{\mp\frac{1}{\mathrm{i}\hbar}x^{\mu}k_{\mu}}\,\mathcal{U}(x,x_0)\circ\tilde{a}_s^{\dagger\,\pm}(\boldsymbol{k})\circ\mathcal{U}^{-1}(x,x_0)
\end{aligned}
\right\}\quad k_0=\sqrt{m^2c^2+\boldsymbol{k}^2},
\tag{6.38}
$$

from which equations (6.36) follow for $x=x_0$. (Notice, the right hand sides of the equations (6.38) are independent of x, due to the Heisenberg relations (2.7).)

From (6.5), (6.6), (6.15) and (6.34)–(6.36), it is clear that all of the obtained expressions for the momentum, charge and spin angular momentum operators in terms of the (invariant) creation/annihilation operators remain unchanged in Heisenberg picture; to obtain a Heisenberg version of these equations, one has formally to add a tilde over the creation/annihilation operators in momentum picture. However, this is not the case with the orbital operator (6.7) because of the presents of derivatives in the integrands in (6.7). We leave to the reader to prove as exercise that, in terms of the operators (6.36), in (6.7) the term $x_{0\,\mu}\mathcal{P}_{\nu}-x_{0\,\nu}\mathcal{P}_{\mu}$, i.e. the first sum in it, should be deleted and tildes over the creation/annihilation operators must be added. Correspondingly, equation (6.7) will read

$$
\begin{aligned}
\tilde{\mathcal{L}}_{\mu\nu} = {}& \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\sum_{s,s'=1}^{3}\int \mathrm{d}^3\boldsymbol{k}\, l_{\mu\nu}^{ss'}(\boldsymbol{k})\bigl\{\tilde{a}_s^{\dagger\,+}(\boldsymbol{k})\circ\tilde{a}_{s'}^{-}(\boldsymbol{k}) - \tilde{a}_s^{\dagger\,-}(\boldsymbol{k})\circ\tilde{a}_{s'}^{+}(\boldsymbol{k})\bigr\}\\
&+\frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))}\sum_{s=1}^{3-\delta_{0m}}\int \mathrm{d}^3\boldsymbol{k}\Bigl\{\tilde{a}_s^{\dagger\,+}(\boldsymbol{k})\Bigl(k_{\mu}\overleftrightarrow{\frac{\partial}{\partial k^{\nu}}} - k_{\nu}\overleftrightarrow{\frac{\partial}{\partial k^{\mu}}}\Bigr)\circ\tilde{a}_s^{-}(\boldsymbol{k})\\
&\qquad - \tilde{a}_s^{\dagger\,-}(\boldsymbol{k})\Bigl(k_{\mu}\overleftrightarrow{\frac{\partial}{\partial k^{\nu}}} - k_{\nu}\overleftrightarrow{\frac{\partial}{\partial k^{\mu}}}\Bigr)\circ\tilde{a}_s^{+}(\boldsymbol{k})\Bigr\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\,.
\end{aligned}
\tag{6.39}
$$

7. The field equations in terms of creation and annihilation operators

For a free vector field (satisfying the Lorenz condition), the equalities (3.17), (3.18), (2.26) and (3.30) form a closed algebraic-functional system of equations for determination of the field operators $\tilde{\mathcal{U}}_\mu$ and $\tilde{\mathcal{U}}_\mu^\dagger$. As these operators and the field's dynamical variables are expressible in terms of the creation and annihilation operators, it is clear that the mentioned system of equations can equivalently be represented in terms of creation and annihilation operators. The derivation of the so-arising system of equations and some its consequences are the main contents of this section.

As a result of (5.6), (5.7) and (5.17) (or (5.18)), we have the decompositions

$$\begin{aligned}\mathcal{U}_\mu &= \sum_{s=1}^{3} \int \mathrm{d}^3\boldsymbol{k} \big(\mathcal{U}_{\mu,s}^{+}(\boldsymbol{k}) + \mathcal{U}_{\mu,s}^{-}(\boldsymbol{k})\big)\\ &= \sum_{s=1}^{3} \int \mathrm{d}^3\boldsymbol{k} \big\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\big\}^{-1/2} \big\{a_s^{+}(\boldsymbol{k}) + a_s^{-}(\boldsymbol{k})\big\} v_\mu^s(\boldsymbol{k})\\ \mathcal{U}_\mu^\dagger &= \sum_{s=1}^{3} \int \mathrm{d}^3\boldsymbol{k} \big(\mathcal{U}_{\mu,s}^{\dagger\,+}(\boldsymbol{k}) + \mathcal{U}_{\mu,s}^{\dagger\,-}(\boldsymbol{k})\big)\\ &= \sum_{s=1}^{3} \int \mathrm{d}^3\boldsymbol{k} \big\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\big\}^{-1/2} \big\{a_s^{\dagger\,+}(\boldsymbol{k}) + a_s^{\dagger\,-}(\boldsymbol{k})\big\} v_\mu^s(\boldsymbol{k}).\end{aligned} \tag{7.1}$$

The definitions of the quantities entering in these equations ensure the fulfillment of the equations (4.7b). Therefore the conditions (4.7a), which are equivalent to (5.8), are the only restrictions on the operators $a_s^{\pm}(\boldsymbol{k})$ and $a_s^{\dagger\,\pm}(\boldsymbol{k})$. Inserting (5.18) into (5.8), multiplying the result by $v^{\mu,s'}(\boldsymbol{k})$, summing over μ, and applying the normalization conditions (4.22), we obtain:

$$[a_s^{\pm}(\boldsymbol{k}), \mathcal{P}_\mu]_{\_} = \mp k_\mu a_s^{\pm}(\boldsymbol{k}) = \mp \sum_{t=1}^{3-\delta_{0m}} \int q_\mu a_s^{\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q})\, \mathrm{d}^3\boldsymbol{q} \tag{7.2a}$$

$$[a_s^{\dagger\,\pm}(\boldsymbol{k}), \mathcal{P}_\mu]_{\_} = \mp k_\mu a_s^{\dagger\,\pm}(\boldsymbol{k}) = \mp \sum_{t=1}^{3-\delta_{0m}} \int q_\mu a_s^{\dagger\,\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q})\, \mathrm{d}^3\boldsymbol{q} \tag{7.2b}$$

$$s = \begin{cases} 1,2,3 & \text{for } m \neq 0\\ 1,2 & \text{for } m = 0\end{cases} \qquad k_0 = \sqrt{m^2c^2+\boldsymbol{k}^2} \qquad q_0 = \sqrt{m^2c^2+\boldsymbol{q}^2}. \tag{7.2c}$$

Substituting in (7.2) the equation (6.5), with integration variable $\boldsymbol{q}$ for $\boldsymbol{k}$ and summation index t for s, we get (do not sum over s!)

$$\begin{aligned}\sum_{t=1}^{3-\delta_{0m}} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + a_t^{\dagger\,-}(\boldsymbol{q}) \circ a_t^{+}(\boldsymbol{q})\big]_{\_}&\\ \pm (1+\tau(\mathcal{U})) a_s^{\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q})\big\}\, \mathrm{d}^3\boldsymbol{q} = 0&\end{aligned} \tag{7.3a}$$

$$\begin{aligned}\sum_{t=1}^{3-\delta_{0m}} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \big\{ \big[a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + a_t^{\dagger\,-}(\boldsymbol{q}) \circ a_t^{+}(\boldsymbol{q})\big]_{\_}&\\ \pm (1+\tau(\mathcal{U})) a_s^{\dagger\,\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q})\big\}\, \mathrm{d}^3\boldsymbol{q} = 0.&\end{aligned} \tag{7.3b}$$

$$s = \begin{cases} 1,2,3 & \text{for } m \neq 0\\ 1,2 & \text{for } m = 0\end{cases}. \tag{7.3c}$$

Consequently, the operators $a_s^{\pm}(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ must be solutions of

$$\big[a_s^{\pm}(\boldsymbol{k}), a_t^{\dagger +}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + a_t^{\dagger -}(\boldsymbol{q}) \circ a_t^{+}(\boldsymbol{q})\big]_{\_} \pm (1+\tau(\mathcal{U})) a_s^{\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) = f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q}) \tag{7.4a}$$

$$\big[a_s^{\dagger\pm}(\boldsymbol{k}), a_t^{\dagger +}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + a_t^{\dagger -}(\boldsymbol{q}) \circ a_t^{+}(\boldsymbol{q})\big]_{\_} \pm (1+\tau(\mathcal{U})) a_s^{\dagger\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) = f_{st}^{\dagger\pm}(\boldsymbol{k},\boldsymbol{q}) \tag{7.4b}$$

$$s,t = \begin{cases} 1,2,3 & \text{for } m \neq 0 \\ 1,2 & \text{for } m = 0 \end{cases}, \tag{7.4c}$$

where $f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q})$ and $f_{st}^{\dagger\pm}(\boldsymbol{k},\boldsymbol{q})$ are (generalized) functions such that

$$\sum_{t=1}^{3-\delta_{0m}} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q})\, \mathrm{d}^3\boldsymbol{q} = 0 \qquad \sum_{t=1}^{3-\delta_{0m}} \int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} f_{st}^{\dagger\pm}(\boldsymbol{k},\boldsymbol{q})\, \mathrm{d}^3\boldsymbol{q} = 0. \tag{7.4d}$$

Since any solution of the field equations (3.17)–(3.18) can be written in the form (7.1) with $a_s^{\pm}(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ being solutions of (7.4) or, equivalently, of (7.3), the system of equations (7.4) or (7.3) is equivalent to the initial system of field equations, consisting of the Klein-Gordon equations (3.17) and the Lorenz conditions (3.18).[24] In this sense, (7.4) or (7.3) is the *system of field equations (of a vector field satisfying the Lorenz condition) in terms of creation and annihilation operators in momentum picture.* If we neglect the polarization indices, this system of equations is identical with the one for an arbitrary free scalar field, obtained in [13]. The reader may also wish to compare (7.4) with a similar system of field equations for a free spinor field, found in [14].

It is important to be mentioned, in the massless case, $m \neq 0$, the field equations (7.4) contain only the polarization modes with $s = 1, 2$ and, consequently, they *do not impose any restrictions on the operators $a_3^{\pm}(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$.* We shall comment on this phenomenon in Sect. 11.

The commutators of the dynamical variables with the momentum operator can easily be found by means of the field equations (7.4). Indeed, from (6.5), (6.6), and (6.15) is evident that for the momentum, charge and spin operators these commutators are expressible as integrals whose integrands are linear combinations of the terms

$$B_{ss'}^{\mu,\mp}(\boldsymbol{q}) := \sum_{t=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} k^\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} [a_s^{\dagger\pm}(\boldsymbol{q}) \circ a_{s'}^{\mp}(\boldsymbol{q}), a_t^{\dagger +}(\boldsymbol{k}) \circ a_t^{-}(\boldsymbol{k}) + a_t^{\dagger -}(\boldsymbol{k}) \circ a_t^{+}(\boldsymbol{k})]_{\_}, \tag{7.5}$$

where $s,s' = \left\{ \begin{smallmatrix} 1,2,3 & \text{if } m\neq 0 \\ 1,2 & \text{if } m=0 \end{smallmatrix} \right.$ for the momentum and charge operators and $s, s, = 1, 2, 3$ for the spin angular momentum operator. Applying the identity $[A \circ B, C]_{\_} \equiv [A, C]_{\_} \circ B + A \circ [B, C]_{\_}$, for operators A, B and C, and (7.3) (which is equivalent to (7.4)), we get (do not sum over s and s'!)

$$\begin{aligned} B_{ss'}^{\mu,\mp}(\boldsymbol{q}) &:= (1+\tau(\mathcal{U})) \sum_{t=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} k^\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} (\mp\delta_{st} \pm \delta_{s't}) \delta^3(\boldsymbol{k}-\boldsymbol{q}) a_s^{\dagger\pm}(\boldsymbol{q}) \circ a_{s'}^{\mp}(\boldsymbol{q}) \\ &= (1+\tau(\mathcal{U})) q^\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \sum_{t=1}^{3-\delta_{0m}} (\mp\delta_{st} \pm \delta_{s't}) a_s^{\dagger\pm}(\boldsymbol{q}) \circ a_{s'}^{\mp}(\boldsymbol{q}) \end{aligned}$$

[24] Recall, the Heisenberg relations (2.7) are incorporated in the constancy of the field operators and are reflected in (4.7a) or (7.2a).

for $s, s' = \left\{ \begin{smallmatrix} 1,2,3 & \text{if } m\neq 0 \\ 1,2 & \text{if } m=0 \end{smallmatrix} \right.$. Obviously, the summation over t in the last expression results in the multiplier $\pm\delta_{ss} \pm \delta_{ss'} = \mp 1 \pm 1 \equiv 0$, and hence

$$B^{\mu,\mp}_{ss'}(\boldsymbol{q}) = 0 \quad \text{for } s, s' = \begin{cases} 1,2,3 & \text{if } m \neq 0 \\ 1,2 & \text{if } m = 0 \end{cases} . \tag{7.6a}$$

Similarly, when $m = 0$ and s or s' is equal to 3, we find

$$\begin{aligned} B^{\mu,\mp}_{ss'}(\boldsymbol{q})\big|_{m=0} &= \pm(1+\tau(\mathcal{U}))q^{\mu}\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} \times \begin{cases} a^{\dagger\,\pm}_3(\boldsymbol{q}) \circ a^{\mp}_{s'}(\boldsymbol{q}) & \text{if } s=3 \text{ and } s'=1,2 \\ -a^{\dagger\,\pm}_s(\boldsymbol{q}) \circ a^{\mp}_3(\boldsymbol{q}) & \text{if } s=1,2 \text{ and } s'=3 \end{cases} \\ &+ (1+\tau(\mathcal{U})) \times \begin{cases} [a^{\dagger\,\pm}_3(\boldsymbol{q}),\, \mathcal{P}^{\mu}]_{\_} \circ a^{\mp}_{s'}(\boldsymbol{q}) & \text{if } s=3 \text{ and } s'=1,2 \\ a^{\dagger\,\pm}_s(\boldsymbol{q}) \circ [a^{\mp}_3(\boldsymbol{q}),\, \mathcal{P}^{\mu}]_{\_} & \text{if } s=1,2 \text{ and } s'=3 \end{cases} . \end{aligned} \tag{7.6b}$$

At last, the case $s = s' = 3$ is insignificant for us as the quantities $B^{\mu,-}_{33}(\boldsymbol{q})$ have a vanishing contribution in (6.15) and (6.7), due to $\sigma^{ss'}_{33}(\boldsymbol{k}) = l^{ss'}_{33}(\boldsymbol{k}) = 0$ (see (6.16) and (6.8)). Now, it is trivial to be seen that (6.5), (6.6), (6.15), and (6.8), on one hand, and (7.6), on another hand, imply the commutation relations

$$[\mathcal{P}_{\mu},\, \mathcal{P}_{\nu}]_{\_} = 0 \tag{7.7}$$

$$[\tilde{\mathcal{Q}},\, \mathcal{P}_{\mu}]_{\_} = 0 \tag{7.8}$$

$$[\tilde{\mathcal{S}}_{\mu\nu},\, \mathcal{P}_{\lambda}]_{\_} = \delta_{0m}\, {}^{S}C^{\lambda}_{\mu\nu} \tag{7.9}$$

$$[\tilde{\mathcal{L}}_{\mu\nu},\, \mathcal{P}_{\lambda}]_{\_} = -\mathrm{i}\hbar\{\eta_{\lambda\mu}\mathcal{P}_{\nu} - \eta_{\lambda\nu}\mathcal{P}_{\mu}\} - \delta_{0m}\, {}^{L}C^{\lambda}_{\mu\nu} \tag{7.10}$$

where

$$\begin{aligned} {}^{S}C^{\lambda}_{\mu\nu} &:= \frac{\mathrm{i}\hbar}{(1+\tau(\mathcal{U}))^2} \sum_{s=1,2} \int \mathrm{d}^3\boldsymbol{k}\sigma^{3s}_{\mu\nu}(\boldsymbol{k})\{B^{\lambda,-}_{3s}(\boldsymbol{k}) - B^{\lambda,+}_{3s}(\boldsymbol{k}) - B^{\lambda,-}_{s3}(\boldsymbol{k}) + B^{\lambda,+}_{s3}(\boldsymbol{k})\}\big|_{m=0} \\ {}^{L}C^{\lambda}_{\mu\nu} &:= \frac{2\mathrm{i}\hbar}{(1+\tau(\mathcal{U}))^2} \sum_{s=1,2} \int \mathrm{d}^3\boldsymbol{k}l^{3s}_{\mu\nu}(\boldsymbol{k})\{B^{\lambda,-}_{3s}(\boldsymbol{k}) - B^{\lambda,+}_{3s}(\boldsymbol{k}) - B^{\lambda,-}_{s3}(\boldsymbol{k}) + B^{\lambda,+}_{s3}(\boldsymbol{k})\}\big|_{m=0} \end{aligned} \tag{7.11}$$

and we have also used (6.16), (6.8) and the equality

$$\begin{aligned} &\sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{q} \sum_{t=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k}k_{\lambda}\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \\ &\qquad \times [a^{\dagger\,\pm}_s(\boldsymbol{q})q_{\mu}\overleftrightarrow{\frac{\partial}{\partial q^{\mu}}} \circ a^{\mp}_s(\boldsymbol{q}), a^{\dagger\,+}_t(\boldsymbol{k}) \circ a^{-}_t(\boldsymbol{k}) + a^{\dagger\,-}_t(\boldsymbol{k}) \circ a^{+}_t(\boldsymbol{k})]_{\_} \\ &\qquad = \pm 2(1+\tau(\mathcal{U}))\eta_{\lambda\nu} \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k}k_{\mu}\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} a^{\dagger\,\pm}_s(\boldsymbol{k}) \circ a^{\mp}_s(\boldsymbol{k}), \end{aligned} \tag{7.12}$$

which can be proved analogously to (7.6a) and which is responsible for the term proportional to $\mathrm{i}\hbar$ in (7.10).

As in Sect 6, the polarization along the 3-momentum, characterized by $s = 3$, is a cause for the appearance of 'abnormal' terms in (7.9) and (7.10) in the massless case, $m = 0$.[25] Combining (7.9), (7.10) and (2.20), we get the commutator between the total angular

[25] The reader may wish to compare (7.9) and (7.10) with similar relations for a free Dirac field in [14] or for a free scalar field in [13] (with $\mathcal{S}_{\mu\nu} = 0$ in the last case). The mentioned abnormal terms destroy also the 'ordinary' commutation relation between the total angular momentum $\mathcal{M}_{\mu\nu} = \mathcal{L}_{\mu\nu} + \mathcal{S}_{\mu\nu}$ and the momentum operator $\mathcal{P}_{\lambda}$; see equation (7.13) below and, e.g., [3, 4, 8, 9].

momentum $\mathcal{M}_{\mu\nu}$ and the momentum operator $\mathcal{P}_\lambda$ as

$$[\mathcal{M}_{\mu\nu},\, \mathcal{P}_\lambda]_{\_} = -\mathrm{i}\hbar\{\eta_{\lambda\mu}\,\mathcal{P}_\nu - \eta_{\lambda\nu}\,\mathcal{P}_\mu\} + \delta_{0m}\{\,{}^{S}C^{\lambda}_{\mu\nu} - {}^{L}C^{\lambda}_{\mu\nu}\}, \tag{7.13}$$

in which the terms mentioned also change the 'ordinary' commutation relation in the massless case. We should also pay attention on the sign before the constant $\mathrm{i}\hbar$ in (7.13) which sign agrees with a similar one in (2.33) and is opposite to the one, usually, accepted in the literature [3, 4, 8, 9].

The expressions for the dynamical variables in momentum picture can be found from equations (7.7)–(7.10) and the general rule (2.4) with $\mathcal{U}(x, x_0)$ being the operator (2.2). However, the spin and orbital angular momentum operators in momentum picture cannot be written, generally, in a closed form for $m = 0$, due to the presents of the terms proportional to δ_{0m} in (7.9) and (7.10). To simplify the situation, below it will be supposed that

$$[\,{}^{S}C^{\lambda}_{\mu\nu},\, \mathcal{P}_\varkappa]_{\_} = 0 \quad [\,{}^{L}C^{\lambda}_{\mu\nu},\, \mathcal{P}_\varkappa]_{\_} = 0. \tag{7.14}$$

If required for some purpose, the reader may generalize, as an exercise, the following results on the base of (2.2) and (2.4) in a case if (7.14) do not hold.

Since (7.8)–(7.10), (7.14) and (2.2) entail (see footnote 6)

$$[\,\tilde{\mathcal{Q}},\, \mathcal{U}(x, x_0)]_{\_} = 0 \tag{7.15}$$

$$[\,\tilde{\mathcal{S}}_{\mu\nu},\, \mathcal{U}(x, x_0)]_{\_} = \delta_{0m}\frac{1}{\mathrm{i}\hbar}(x_\lambda - x_{0\,\lambda})\,{}^{S}C^{\lambda}_{\mu\nu} \circ \mathcal{U}(x, x_0) \tag{7.16}$$

$$\begin{aligned}[\,\tilde{\mathcal{L}}_{\mu\nu},\, \mathcal{U}(x, x_0)]_{\_} = &-\{(x_\mu - x_{0\,\mu})\,\mathcal{P}_\nu - (x_\nu - x_{0\,\nu})\,\mathcal{P}_\mu\} \circ \mathcal{U}(x, x_0) \\ &- \delta_{0m}\frac{1}{\mathrm{i}\hbar}(x_\lambda - x_{0\,\lambda})\,{}^{L}C^{\lambda}_{\mu\nu} \circ \mathcal{U}(x, x_0),\end{aligned} \tag{7.17}$$

by virtue of $\mathcal{A}(x) = \tilde{\mathcal{A}}(x) - [\,\tilde{\mathcal{A}}(x),\, \mathcal{U}(x, x_0)]_{\_} \circ \mathcal{U}^{-1}(x, x_0)$ (see (2.4)), it follows that the charge, spin and orbital angular momentum operators in *momentum picture* respectively are:

$$\mathcal{Q} = \tilde{\mathcal{Q}} \tag{7.18}$$

$$\mathcal{S}_{\mu\nu} = \tilde{\mathcal{S}}_{\mu\nu} - \delta_{0m}\frac{1}{\mathrm{i}\hbar}(x_\lambda - x_{0\,\lambda})\,{}^{S}C^{\lambda}_{\mu\nu}. \tag{7.19}$$

$$\mathcal{L}_{\mu\nu} = \tilde{\mathcal{L}}_{\mu\nu} + (x_\mu - x_{0\,\mu})\,\mathcal{P}_\nu - (x_\nu - x_{0\,\nu})\,\mathcal{P}_\mu + \delta_{0m}\frac{1}{\mathrm{i}\hbar}(x_\lambda - x_{0\,\lambda})\,{}^{L}C^{\lambda}_{\mu\nu}. \tag{7.20}$$

Explicitly, by virtue of (6.7), the orbital angular momentum operator is

$$\begin{aligned}\tilde{\mathcal{L}}_{\mu\nu} = &\frac{1}{1+\tau(\mathcal{U})}\sum_{s=1}^{3-\delta_{0m}}\int \mathrm{d}^3\boldsymbol{k}(x_\mu k_\nu - x_\nu k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\{a_s^{\dagger\,+}(\boldsymbol{k})\circ a_s^{-}(\boldsymbol{k}) + a_s^{\dagger\,-}(\boldsymbol{k})\circ a_s^{+}(\boldsymbol{k})\} \\ &+ \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\sum_{s,s'=1}^{3}\int \mathrm{d}^3\boldsymbol{k}\, l^{ss'}_{\mu\nu}(\boldsymbol{k})\{a_s^{\dagger\,+}(\boldsymbol{k})\circ a_{s'}^{-}(\boldsymbol{k}) - a_s^{\dagger\,-}(\boldsymbol{k})\circ a_{s'}^{+}(\boldsymbol{k})\} \\ &+ \frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))}\sum_{s=1}^{3-\delta_{0m}}\int \mathrm{d}^3\boldsymbol{k}\Big\{a_s^{\dagger\,+}(\boldsymbol{k})\Big(k_\mu\frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu\frac{\overleftrightarrow{\partial}}{\partial k^\mu}\Big)\circ a_s^{-}(\boldsymbol{k}) \\ &\qquad - a_s^{\dagger\,-}(\boldsymbol{k})\Big(k_\mu\frac{\overleftrightarrow{\partial}}{\partial k^\nu} - k_\nu\frac{\overleftrightarrow{\partial}}{\partial k^\mu}\Big)\circ a_s^{+}(\boldsymbol{k})\Big\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}},\end{aligned} \tag{7.21}$$

As we see again, these results differ in the massless case from the ones, expected from the outcome of [13, 14], by terms depending on the operators $a_3^{\pm}(\boldsymbol{k})$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})$ with polarization variable $s = 3$.

8. The commutation relations

Comparing the field equations (7.4) with similar ones for an arbitrary free scalar field, obtained in [13], we see that the only difference between them is that the creation and annihilation operators depend on the polarization indices in the vector field case, which indices are missing when scalar fields are concerned. It is a simple observation, a polarization variable, say s, is coupled always to a momentum variable, say $\boldsymbol{k}$, and can be considered as its counterpart. This allows s and $\boldsymbol{k}$ to be treated on equal footing in order that one takes into account that $\boldsymbol{k} \in \mathbb{R}^3$ is a continuous variable, while s is a discrete one, taking the values $s = 1, 2, 3$ for a massive vector field and $s = 1, 2$ for a massless vector field satisfying the Lorenz condition. Therefore the transformations

$$\begin{gathered}
\boldsymbol{k} \mapsto (s, \boldsymbol{k}) \qquad \varphi_0^{\pm}(\boldsymbol{k}) \mapsto a_s^{\pm}(\boldsymbol{k}) \qquad \varphi_0^{\dagger\,\pm}(\boldsymbol{k}) \mapsto a_s^{\dagger\,\pm}(\boldsymbol{k}) \\
\int \mathrm{d}^3\boldsymbol{k} \mapsto \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} \qquad \delta^3(\boldsymbol{k}-\boldsymbol{q}) \mapsto \delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q}),
\end{gathered} \tag{8.1}$$

where $\varphi_0^{\pm}(\boldsymbol{k})$ and $\varphi_0^{\dagger\,\pm}(\boldsymbol{k})$ are the creation/annihilation operators for a free scalar field, allow us to transfer automatically all results regarding the field equations of a free scalar field to free vector field (satisfying the Lorenz condition). The same conclusion is, evidently valid and with respect to results in which the momentum and charge operators are involved.[26] As a particular realization of these assertions, the commutation relations for a free vector field (satisfying the Lorenz condition) will be considered below; in other words, a second quantization of such a field will be performed by their means. The reader can find a motivation for an introduction of these relations in books like [1–3, 16].

Before writing the commutation relations for a free vector field satisfying the Lorenz condition, we would like to state explicitly the *additional to Lagrangian formalism conditions*, imposed on the field operators, which *reduce* the field equations (7.4) to these relations. As a *first* conditions, it is supposed the (anti)commutators between all creation and/or annihilation operators to be *c*-numbers, i.e. to be proportional to the identity mapping $\mathsf{id}_{\mathcal{F}}$ of the system's (field's) Hilbert space $\mathcal{F}$ of states. This hypothesis reduces the field equations (7.4) to a certain algebraic-functional system of equations which can be obtained from a similar one for a scalar field, derived in [13], by means of the rules (8.1). As a *second* restriction, it is demanded the last system of equations to be an identity with respect to the creation and annihilation operators. A consequence of this restriction is that the (anti)commutators between creation and/or annihilation operators are uniquely defined as operators proportional to $\mathsf{id}_{\mathcal{F}}$.[27] At this stage of the theory's development it remains undetermined whether a vector field should be quantize via commutators or anticommutators; the Lagrangian (3.7), we started off, is insensitive with respect to that choice. To be achieved a conformity with the experimental data, one should choose, as a *third* additional restriction, a quantization via commutators, not via anticommutators.[28, 29]

[26] However, when the angular momentum operator is concerned, one should be quite careful as the changes (8.1) will produce (6.7) with missing the integral depending on $l_{\mu\nu}^{ss'}(\boldsymbol{k})$. Moreover, the rules (8.1) cannot be applied at all to results in which the spin is involved; e.g. they will produce identically vanishing spin angular momentum operator of free vector fields instead of the expression (6.15).

[27] The mentioned system of equations does not give any information about the operators $a_s^{\pm}(\boldsymbol{0})|_{m=0}$ and $a_s^{\dagger\,\pm}(\boldsymbol{0})|_{m=0}$, which describe massless particles with vanishing 4-momentum and, possibly, non-vanishing charge and spin. By convention, we assume these operators to satisfy the same (anti)commutation relations as the creation/annihilation operators for $(\boldsymbol{k}, m) \neq (\boldsymbol{0}, 0)$.

[28] Equivalently, one may demand a charge symmetry of the theory, the validity of the spin-statistics theorem, etc.

[29] This condition can be incorporated in the Lagrangian formalism by a suitable choice of a Lagrangian. It follows from the Lagrangian (3.7), we started off, if the field considered is neutral/Hermitian. For details, see Sect. 12.

As a result of the described additional hypotheses, the field equations (7.4) reduce to the following system of commutation relations, which is obtainable from a similar one for a free scalar field, derived in [13], via the changes (8.1):

$$
\begin{aligned}
&[a_s^{\pm}(\boldsymbol{k}), a_t^{\pm}(\boldsymbol{q})]_{\_} = 0 && [a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{\dagger\,\pm}(\boldsymbol{q})]_{\_} = 0 \\
&[a_s^{\mp}(\boldsymbol{k}), a_t^{\pm}(\boldsymbol{q})]_{\_} = \pm\tau(\mathcal{U})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})\,\mathsf{id}_{\mathcal{F}} && [a_s^{\dagger\,\mp}(\boldsymbol{k}), a_t^{\dagger\,\pm}(\boldsymbol{q})]_{\_} = \pm\tau(\mathcal{U})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})\,\mathsf{id}_{\mathcal{F}} \\
&[a_s^{\pm}(\boldsymbol{k}), a_t^{\dagger\,\pm}(\boldsymbol{q})]_{\_} = 0 && [a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{\pm}(\boldsymbol{q})]_{\_} = 0 \\
&[a_s^{\mp}(\boldsymbol{k}), a_t^{\dagger\,\pm}(\boldsymbol{q})]_{\_} = \pm\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})\,\mathsf{id}_{\mathcal{F}} && [a_s^{\dagger\,\mp}(\boldsymbol{k}), a_t^{\pm}(\boldsymbol{q})]_{\_} = \pm\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})\,\mathsf{id}_{\mathcal{F}}
\end{aligned}
\tag{8.2}
$$

where, as it was said above, the values of the polarization indices depend on the mass parameter m according to

$$
s,t = \begin{cases} 1,2,3 & \text{for } m \neq 0 \\ 1,2 & \text{for } m = 0 \end{cases}, \tag{8.3}
$$

the zero operator of $\mathcal{F}$ is denoted by 0, and $\tau(\mathcal{U})$ takes care of is the field neutral/Hermitian ($\mathcal{U}^\dagger = \mathcal{U}$, $\tau(\mathcal{U}) = 1$) or charged/non-Hermitian ($\mathcal{U}^\dagger \neq \mathcal{U}$, $\tau(\mathcal{U}) = 0$) and ensures correct commutation relations in the Hermitian case, when $a_s^{\dagger\,\pm}(\boldsymbol{k}) = a_s^{\pm}(\boldsymbol{k})$.

Let us emphasize once again, the commutation relations (8.2) are equivalent to the field equations (7.4) and, consequently, to the initial system of equations (3.17)–(3.18) under the made hypotheses. If by some reason one or more of these additional to the Lagrangian formalism conditions is rejected, the trilinear system of equations (7.4), which is more general than (8.2), should be considered.

A feature of (8.2) is that, in the massless case, the operators $a_3^{\pm}(\boldsymbol{k})$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})$, i.e. the polarization modes with $s = 3$ (along the vector $\boldsymbol{k}$), do not enter in it and hence, these operators remain completely arbitrary. In that sense, these modes remain not 'second quantized', i.e. the Lagrangian formalism does not give any information about the (anti)commutation relations between themselves or between them and other creation and annihilation operators.

As we have noted in [13], the concepts of a distribution (generalized function) and operator-valued distribution appear during the derivation of the commutation relations (8.2). In particular, the canonical commutation relations (8.2) have a sense iff the commutators of the creation and/or annihilation operators are operator-valued distributions (proportional to $\mathsf{id}_{\mathcal{F}}$), which is *not* the case if the fields considered are described via ordinary operators acting on $\mathcal{F}$. These facts point to inherent contradiction of quantum field theory if the field variables are considered as operators acting on a Hilbert space. The rigorous mathematical setting requires the fields variables to be regarded as operator-valued distributions. However, such a setting is out of the scope of the present work and the reader is referred to books like [8, 9, 24, 25] for more details and realization of that program. In what follows, the distribution character of the quantum fields will be encoded in the Dirac's delta function, which will appear in relations like (7.4) and (8.2).

As an application of the commutation relations (8.2), we shall calculate the commutators between the components of the spin operator and between them and the charge operator. For the purpose, we shall apply the following commutation relations between quadratic com-

binations of creation and/or annihilation operators:

$$
\begin{aligned}
&[a_s^{\dagger\pm}(\boldsymbol{k})\circ a_{s'}^{\mp}(\boldsymbol{k}),a_t^{\dagger\pm}(\boldsymbol{p})\circ a_{t'}^{\mp}(\boldsymbol{p})]_{\_}=\{\mp\delta_{st'}a_t^{\dagger\pm}(\boldsymbol{p})\circ a_{s'}^{\mp}(\boldsymbol{k})\pm\delta_{s't}a_s^{\dagger\pm}(\boldsymbol{k})\circ a_{t'}^{\mp}(\boldsymbol{p})\}\delta^3(\boldsymbol{k}-\boldsymbol{p})\\
&[a_s^{\dagger\pm}(\boldsymbol{k})\circ a_{s'}^{\mp}(\boldsymbol{k}),a_t^{\mp}(\boldsymbol{p})\circ a_{t'}^{\dagger\pm}(\boldsymbol{p})]_{\_}=\{\mp\delta_{st}a_{s'}^{\mp}(\boldsymbol{k})\circ a_{t'}^{\dagger\pm}(\boldsymbol{p})\pm\delta_{s't'}a_t^{\mp}(\boldsymbol{p})\circ a_s^{\dagger\pm}(\boldsymbol{k})\}\delta^3(\boldsymbol{k}-\boldsymbol{p})\\
&[a_s^{\pm}(\boldsymbol{k})\circ a_{s'}^{\dagger\mp}(\boldsymbol{k}),a_t^{\pm}(\boldsymbol{p})\circ a_{t'}^{\dagger\mp}(\boldsymbol{p})]_{\_}=\{\mp\delta_{st'}a_t^{\pm}(\boldsymbol{p})\circ a_{s'}^{\dagger\mp}(\boldsymbol{k})\pm\delta_{s't}a_s^{\pm}(\boldsymbol{k})\circ a_{t'}^{\dagger\mp}(\boldsymbol{p})\}\delta^3(\boldsymbol{k}-\boldsymbol{p})\\
&[a_s^{\dagger\pm}(\boldsymbol{k})\circ a_{s'}^{\mp}(\boldsymbol{k}),a_t^{\dagger\mp}(\boldsymbol{p})\circ a_{t'}^{\pm}(\boldsymbol{p})]_{\_}=\tau(\mathcal{U})\{\mp\delta_{st}a_{s'}^{\mp}(\boldsymbol{k})\circ a_{t'}^{\pm}(\boldsymbol{p})\pm\delta_{s't'}a_t^{\dagger\mp}(\boldsymbol{p})\circ a_s^{\dagger\pm}(\boldsymbol{k})\}\delta^3(\boldsymbol{k}-\boldsymbol{p})\\
&[a_s^{\dagger\pm}(\boldsymbol{k})\circ a_{s'}^{\mp}(\boldsymbol{k}),a_t^{\pm}(\boldsymbol{p})\circ a_{t'}^{\dagger\mp}(\boldsymbol{p})]_{\_}=\tau(\mathcal{U})\{\mp\delta_{st'}a_t^{\pm}(\boldsymbol{p})\circ a_{s'}^{\mp}(\boldsymbol{k})\pm\delta_{s't}a_s^{\dagger\pm}(\boldsymbol{k})\circ a_{t'}^{\dagger\mp}(\boldsymbol{p})\}\delta^3(\boldsymbol{k}-\boldsymbol{p})\\
&[a_s^{\pm}(\boldsymbol{k})\circ a_{s'}^{\dagger\mp}(\boldsymbol{k}),a_t^{\mp}(\boldsymbol{p})\circ a_{t'}^{\dagger\pm}(\boldsymbol{p})]_{\_}=\tau(\mathcal{U})\{\mp\delta_{st}a_{s'}^{\dagger\mp}(\boldsymbol{k})\circ a_{t'}^{\dagger\pm}(\boldsymbol{p})\pm\delta_{s't'}a_t^{\mp}(\boldsymbol{p})\circ a_s^{\pm}(\boldsymbol{k})\}\delta^3(\boldsymbol{k}-\boldsymbol{p}),
\end{aligned}
\tag{8.4}
$$

where the polarization indices s, s', t, and t' take the values 1, 2 and 3 for $m\neq 0$ and 1 and 2 for $m=0$. These equalities are simple corollaries of the identities $[A,B\circ C]_{\_}=[A,B]_{\_}\circ C+B\circ[A,C]_{\_}$ and $[B\circ C,A]_{\_}=[B,A]_{\_}\circ C+B\circ[C,A]_{\_}$, applied in this order to the left-hand-sides of (8.4), and (8.2).

Applying (6.15), (7.19) and (8.4), we find:

$$
\begin{aligned}
&[\tilde{\mathcal{S}}_{\mu\nu},\tilde{\mathcal{S}}_{\varkappa\lambda}]_{\_}=[\mathcal{S}_{\mu\nu},\mathcal{S}_{\varkappa\lambda}]_{\_}\big|_{m\neq0}=\frac{\hbar^2}{(1+\tau(\mathcal{U}))^2}\\
&\times\sum_{s,s',t=1}^{3-\delta_{0m}}\int\mathrm{d}^3\boldsymbol{k}\big\{\big(\sigma_{\varkappa\lambda}^{ss'}(\boldsymbol{k})\sigma_{\mu\nu}^{s't}(\boldsymbol{k})-\sigma_{\mu\nu}^{ss'}(\boldsymbol{k})\sigma_{\varkappa\lambda}^{s't}(\boldsymbol{k})\big)\big(a_s^{\dagger+}(\boldsymbol{k})\circ a_t^{-}(\boldsymbol{k})-a_s^{\dagger-}(\boldsymbol{k})\circ a_t^{+}(\boldsymbol{k})\big)\\
&\qquad\qquad+\tau(\mathcal{U})\sigma_{\mu\nu}^{ss'}(\boldsymbol{k})\sigma_{\varkappa\lambda}^{s't}(\boldsymbol{k})\big(a_s^{-}(\boldsymbol{k})\circ a_t^{+}(\boldsymbol{k})-a_s^{+}(\boldsymbol{k})\circ a_t^{-}(\boldsymbol{k})\big)\\
&+\tau(\mathcal{U})\sigma_{\mu\nu}^{ts'}(\boldsymbol{k})\sigma_{\varkappa\lambda}^{ss'}(\boldsymbol{k})\big(a_s^{\dagger+}(\boldsymbol{k})\circ a_t^{\dagger-}(\boldsymbol{k})-a_s^{\dagger-}(\boldsymbol{k})\circ a_t^{\dagger+}(\boldsymbol{k})\big)\big\}+\delta_{0m}\tilde{f}_{\mu\nu\varkappa\lambda}(a_3^{\pm},a_3^{\dagger\pm}),
\end{aligned}
\tag{8.5}
$$

where $\tilde{f}_{\mu\nu\varkappa\lambda}(a_3^{\pm},a_3^{\dagger\pm})$ is a term whose integrand is a homogeneous expression in $a_3^{\pm}(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$ and which term is set equal to zero for $m\neq0$. The summation over s' in (8.5) can be performed explicitly by means of (6.16):

$$
\sum_{s'=1}^{3-\delta_{0m}}\big\{\sigma_{\mu\nu}^{ss'}(\boldsymbol{k})\sigma_{\varkappa\lambda}^{s't}(\boldsymbol{k})\big\}=v_{\nu\varkappa}(\boldsymbol{k})v_\mu^s(\boldsymbol{k})v_\lambda^t(\boldsymbol{k})-(\mu\leftrightarrow\nu)-(\varkappa\leftrightarrow\lambda)
\tag{8.6}
$$

$$
\sum_{s'=1}^{3-\delta_{0m}}\big\{\sigma_{\mu\nu}^{ss'}(\boldsymbol{k})\sigma_{\varkappa\lambda}^{s't}(\boldsymbol{k})-\sigma_{\varkappa\lambda}^{ss'}(\boldsymbol{k})\sigma_{\mu\nu}^{s't}(\boldsymbol{k})\big\}=-v_{\mu\varkappa}(\boldsymbol{k})\sigma_{\nu\lambda}^{st}(\boldsymbol{k})-(\mu\leftrightarrow\nu)-(\varkappa\leftrightarrow\lambda),
\tag{8.7}
$$

where

$$
v_{\mu\nu}(\boldsymbol{k}):=\sum_{s=1}^{3-\delta_{0m}}v_\mu^s(\boldsymbol{k})v_\nu^s(\boldsymbol{k})
\tag{8.8}
$$

is given via (4.26) and (4.27) and the symbol $-(\mu\leftrightarrow\nu)$ means that we have to subtract the previous terms by making the change $\mu\leftrightarrow\nu$, i.e. an antisymmetrization over μ and ν must be performed.

Similar calculations, based on (6.15) and (6.6), show that[30]

$$
\begin{aligned}
[\tilde{\mathcal{S}}_{\mu\nu},\tilde{\mathcal{Q}}]_{\_}=\frac{\mathrm{i}\hbar q\tau(\mathcal{U})}{1+\tau(\mathcal{U})}\sum_{s,t=1}^{3-\delta_{0m}}\int\mathrm{d}^3\boldsymbol{k}\sigma_{\mu\nu}^{ts}(\boldsymbol{k})\big\{a_s^{-}(\boldsymbol{k})\circ a_t^{+}(\boldsymbol{k})-a_s^{+}(\boldsymbol{k})\circ a_t^{-}(\boldsymbol{k})&\\
+a_s^{\dagger+}(\boldsymbol{k})\circ a_t^{\dagger-}(\boldsymbol{k})-a_s^{\dagger-}(\boldsymbol{k})\circ a_t^{\dagger+}(\boldsymbol{k})\big\}+\delta_{0m}\tilde{f}_{\mu\nu}(a_3^{\pm},a_3^{\dagger\pm}),&
\end{aligned}
$$

[30] This result can formally be obtained from (8.5) with $-\frac{1}{\mathrm{i}\hbar}(1+\tau(\mathcal{U}))q\delta^{st}$ for $\sigma_{\varkappa\lambda}^{st}(\boldsymbol{k})$.

where $\tilde{f}_{\mu\nu}(a_3^{\pm}, a_3^{\dagger\pm})$ is a term whose integrand is a homogeneous expression in $a_3^{\pm}(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$ and which term is set equal to zero for $m \neq 0$. Hence, recalling that $q = 0$ for a Hermitian field and $\tau(\mathcal{U}) = 0$ for a non-Hermitian one and, consequently, $q\tau(\mathcal{U}) \equiv 0$, we get

$$[\tilde{\mathcal{S}}_{\mu\nu},\, \tilde{\mathcal{Q}}]_{\_} = \delta_{0m}\tilde{f}_{\mu\nu}(a_3^{\pm}, a_3^{\dagger\pm}) \tag{8.9}$$

in Heisenberg picture. Therefore, we obtain

$$[\mathcal{S}_{\mu\nu},\, \mathcal{Q}]_{\_} = \delta_{0m} f_{\mu\nu}(a_3^{\pm}, a_3^{\dagger\pm}) \tag{8.10}$$

in momentum picture (see (2.4)). In particular, we have

$$[\mathcal{S}_{\mu\nu},\, \mathcal{Q}]_{\_} = 0 \qquad \text{if } m \neq 0 \text{ or if } m = 0 \text{ and } a_3^{\pm}(\boldsymbol{k}) = a_3^{\dagger\pm}(\boldsymbol{k}) = 0; \tag{8.11}$$

so, for a massive vector field, the spin and charge operators commute.

As other corollary of (8.2), we shall establish the equations

$$[\mathcal{U}_\lambda,\, \mathcal{M}_{\mu\nu}(x, x_0)]_{\_} = x_\mu[\mathcal{U}_\lambda,\, \mathcal{P}_\nu]_{\_} - x_\nu[\mathcal{U}_\lambda,\, \mathcal{P}_\mu]_{\_} + \mathrm{i}\hbar(\mathcal{U}_\mu\eta_{\nu\lambda} - \mathcal{U}_\nu\eta_{\mu\lambda}) \tag{8.12a}$$

$$[\mathcal{U}_\lambda^\dagger,\, \mathcal{M}_{\mu\nu}(x, x_0)]_{\_} = x_\mu[\mathcal{U}_\lambda^\dagger,\, \mathcal{P}_\nu]_{\_} - x_\nu[\mathcal{U}_\lambda^\dagger,\, \mathcal{P}_\mu]_{\_} + \mathrm{i}\hbar(\mathcal{U}_\mu^\dagger\eta_{\nu\lambda} - \mathcal{U}_\nu^\dagger\eta_{\mu\lambda}), \tag{8.12b}$$

where $m \neq 0$, which are part of the conditions ensuring the relativistic covariance of the theory considered [15]. The condition $m \neq 0$ is essential one. For $m = 0$, additional terms, depending on $a_s^{\pm}(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$, $s = 1, 2, 3$, should be added to the right hand sides of (8.12); these terms are connected with the gauge symmetry of the massless case — for some details about that situation for an electromagnetic field, see [15, § 8.4].

We shall prove the equalities

$$[\mathcal{U}_\lambda,\, \mathcal{S}_{\mu\nu}(x, x_0)]_{\_} = \mathrm{i}\hbar\int \mathrm{d}^3\boldsymbol{p}\Big\{\sum_{t=1}^{3} v_\lambda^t(\boldsymbol{p})v_\mu^t(\boldsymbol{p})\big(\mathcal{U}_\nu^+(\boldsymbol{p}) + \mathcal{U}_\nu^-(\boldsymbol{p})\big) - (\mu \leftrightarrow \nu)\Big\} \tag{8.13a}$$

$$\begin{aligned}[\mathcal{U}_\lambda,\, \mathcal{L}_{\mu\nu}(x, x_0)]_{\_} = & x_\mu[\mathcal{U}_\lambda,\, \mathcal{P}_\nu]_{\_} - x_\nu[\mathcal{U}_\lambda,\, \mathcal{P}_\mu]_{\_} - \mathrm{i}\hbar\int \mathrm{d}^3\boldsymbol{p}\Big\{\Big(\eta_{\lambda\mu} + \sum_{t=1}^{3} v_\lambda^t(\boldsymbol{p})v_\mu^t(\boldsymbol{p})\Big) \\ & \times \big(\mathcal{U}_\nu^+(\boldsymbol{p}) + \mathcal{U}_\nu^-(\boldsymbol{p})\big) - (\mu \leftrightarrow \nu)\Big\} \qquad \text{for } m \neq 0\end{aligned} \tag{8.13b}$$

and similar ones with $\mathcal{U}_\lambda^\dagger$ for $\mathcal{U}_\lambda$, from which the equations (8.12) immediately follow, due to (5.6), (5.7) and (3.24). Here and below $p_0 := \sqrt{m^2c^2 + \boldsymbol{p}^2}$ and the symbol $-(\mu \leftrightarrow \nu)$ means that we have to subtract the previous terms by making the change $\mu \leftrightarrow \nu$, i.e. the previous expression has to be antisymmetrized relative to μ and ν.

Equation (8.13a) is a simple corollary of (7.1), (8.2) and (6.16). To derive (8.13b), we substitute (7.1) and (6.7) in its l.h.s. and then, after an integration by parts of the terms proportional to $\frac{\partial a_s^{\pm}(\boldsymbol{p})}{\partial p^\nu}$ and using (6.5) and (6.8), we obtain

$$\begin{aligned}[\mathcal{U}_\lambda,\, \mathcal{L}_{\mu\nu}(x, x_0)]_{\_} = {} & x_\mu[\mathcal{U}_\lambda,\, \mathcal{P}_\nu]_{\_} - x_\nu[\mathcal{U}_\lambda,\, \mathcal{P}_\mu]_{\_} + \mathrm{i}\hbar\sum_{s=1}^{3}\int \mathrm{d}^3\boldsymbol{p}\{2c(2\pi\hbar)^3\sqrt{m^2c^2 + \boldsymbol{p}^2}\}^{1/2} \\ & \times \Big\{\Big(\eta_{\lambda\sigma} + \sum_{t=1}^{3} v_\lambda^t(\boldsymbol{p})v_\sigma^t(\boldsymbol{p})\Big)p_\mu\frac{\partial v^{\sigma,s}(\boldsymbol{p})}{\partial p^\nu}\big(a_s^+(\boldsymbol{p}) + a_s^-(\boldsymbol{p})\big) - (\mu \leftrightarrow \nu)\Big\}.\end{aligned}$$

Since from (4.26a) with $m \neq 0$ [31] and (4.19) it follows

$$\Big(\eta_{\lambda\sigma} + \sum_{t=1}^{3} v_\lambda^t(\boldsymbol{p}) v_\sigma^t(\boldsymbol{p})\Big) p_\mu \frac{\partial v^{\sigma,s}(\boldsymbol{p})}{\partial p^\nu} = \frac{1}{m^2c^2} p_\lambda p_\sigma p_\mu \frac{\partial v^{\sigma,s}(\boldsymbol{p})}{\partial p^\nu}$$
$$= -\frac{1}{m^2c^2} p_\lambda p_\mu \eta_{\sigma\nu} v^{\sigma,s}(\boldsymbol{p}) = -\Big(\eta_{\lambda\mu} + \sum_{t=1}^{3} v_\lambda^t(\boldsymbol{p}) v_\mu^t(\boldsymbol{p})\Big) v_\nu^s(\boldsymbol{p}) \qquad \text{for } m \neq 0,$$

the last equality implying (8.13b), due to (5.17) and (5.5). Q.E.D.

As it was said above, the relations (8.12) are not valid for $m = 0$ in the theory considered, unless some additional terms are taken into account. This fact is connected with the quantization method adopted in the present work for massless vector fields. Other such methods may restore the validity of (8.12) for $m = 0$; for example, such is the Gupta-Bleuler quantization of electromagnetic field [1,6], as it is proved in [6, § 19.1] (for interacting electromagnetic and spin $\frac{1}{2}$ fields).

An interesting result is that equation (8.12), regardless of the condition $m \neq 0$, implies the relation

$$[\mathcal{M}_{\varkappa\lambda}, \mathcal{M}_{\mu\nu}]_{\_} = -\mathrm{i}\hbar\{\eta_{\varkappa\mu}\mathcal{M}_{\lambda\nu} - \eta_{\lambda\mu}\mathcal{M}_{\varkappa\nu} - \eta_{\varkappa\nu}\mathcal{M}_{\lambda\mu} + \eta_{\lambda\nu}\mathcal{M}_{\varkappa\mu}\}. \tag{8.14}$$

To prove this, we notice that (8.12), in momentum representation in Heisenberg picture, is equivalent to (see (6.34))

$$\begin{aligned}
[\tilde{\mathcal{U}}_\lambda^{\pm}(\boldsymbol{k}), \tilde{\mathcal{M}}_{\mu\nu}]_{_} &= \mathrm{i}\hbar\Big(k_\mu\frac{\partial}{\partial k^\nu} - k_\nu\frac{\partial}{\partial k^\mu}\Big)\tilde{\mathcal{U}}_\lambda^{\pm}(\boldsymbol{k}) + \mathrm{i}\hbar\big(\tilde{\mathcal{U}}_\mu^{\pm}(\boldsymbol{k})\eta_{\nu\lambda} - \tilde{\mathcal{U}}_\nu^{\pm}(\boldsymbol{k})\eta_{\mu\lambda}\big)\\
[\tilde{\mathcal{U}}_\lambda^{\dagger\pm}(\boldsymbol{k}), \tilde{\mathcal{M}}_{\mu\nu}]_{_} &= \mathrm{i}\hbar\Big(k_\mu\frac{\partial}{\partial k^\nu} - k_\nu\frac{\partial}{\partial k^\mu}\Big)\tilde{\mathcal{U}}_\lambda^{\dagger\pm}(\boldsymbol{k}) + \mathrm{i}\hbar\big(\tilde{\mathcal{U}}_\mu^{\dagger\pm}(\boldsymbol{k})\eta_{\nu\lambda} - \tilde{\mathcal{U}}_\nu^{\dagger\pm}(\boldsymbol{k})\eta_{\mu\lambda}\big),
\end{aligned} \tag{8.15}$$

where $k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2}$. Now, applying (6.3), (6.14) and the identity $[A \circ B, C]_{\_} = A \circ [B, C]_{\_} + [A, C]_{\_} \circ B$, one can prove, after a trivial but long calculations, that

$$\begin{aligned}
[\tilde{\mathcal{S}}_{\varkappa\lambda}, \tilde{\mathcal{M}}_{\mu\nu}]_{_} &= -\mathrm{i}\hbar\{\eta_{\varkappa\mu}\tilde{\mathcal{S}}_{\lambda\nu} - \eta_{\lambda\mu}\tilde{\mathcal{S}}_{\varkappa\nu} - \eta_{\varkappa\nu}\tilde{\mathcal{S}}_{\lambda\mu} + \eta_{\lambda\nu}\tilde{\mathcal{S}}_{\varkappa\mu}\}\\
[\tilde{\mathcal{L}}_{\varkappa\lambda}, \tilde{\mathcal{M}}_{\mu\nu}]_{_} &= -\mathrm{i}\hbar\{\eta_{\varkappa\mu}\tilde{\mathcal{L}}_{\lambda\nu} - \eta_{\lambda\mu}\tilde{\mathcal{L}}_{\varkappa\nu} - \eta_{\varkappa\nu}\tilde{\mathcal{L}}_{\lambda\mu} + \eta_{\lambda\nu}\tilde{\mathcal{L}}_{\varkappa\mu}\},
\end{aligned} \tag{8.16}$$

from where equation (8.14) follows in Heisenberg picture as $\tilde{\mathcal{M}}_{\mu\nu} = \tilde{\mathcal{L}}_{\mu\nu} + \tilde{\mathcal{S}}_{\mu\nu}$. Notice, this derivation of (8.14) demonstrates that (8.14) is a consequence of the validity of (8.12) regardless of the fulfillment of the commutation relations (8.2). Similarly, equations (8.18) imply (8.19) regardless of the validity of (8.2).

By virtue of the identity $[A, B \circ C]_{\_} = [A, B]_{\_} \circ C + B \circ [A, C]_{\_}$, the relations

$$[a_s^{\pm}(\boldsymbol{k}), \mathcal{Q}]_{\_} = q a_s^{\pm}(\boldsymbol{k}) \quad [a_s^{\dagger\pm}(\boldsymbol{k}), \mathcal{Q}]_{\_} = -q a_s^{\dagger\pm}(\boldsymbol{k}) \qquad s = \left\{ \begin{smallmatrix} 1,2,3 & \text{if } m\neq0 \\ 1,2 & \text{if } m=0 \end{smallmatrix} \right. \tag{8.17}$$

are trivial corollaries from (6.6) and the commutation relations (8.2). From here and (7.1), we get

$$[\mathcal{U}_\mu, \mathcal{Q}]_{\_} = q\mathcal{U}_\mu + \delta_{0m}(\cdots) \quad [\mathcal{U}_\mu^\dagger, \mathcal{Q}]_{\_} = -q\mathcal{U}_\mu^\dagger + \delta_{0m}(\cdots)_\dagger, \tag{8.18}$$

with $(\cdots)$ and $(\cdots)_\dagger$ denoting expressions which are linear and homogeneous in $a_3^{\pm}(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$. So, the equations (3.34) are consequences of the Lagrangian formalism under consideration if $m \neq 0$ or if $m = 0$ and $a_3^{\pm}(\boldsymbol{k}) = a_3^{\dagger\pm}(\boldsymbol{k}) = 0$. Moreover, the relations (8.17) entail the commutativity of bilinear functions/functionals of $a_s^{\pm}(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$, $s = \left\{ \begin{smallmatrix} 1,2,3 & \text{if } m\neq0 \\ 1,2 & \text{if } m=0 \end{smallmatrix} \right.$, with the charge operator $\mathcal{Q}$. In particular, we have (see (6.5)–(6.7) and (6.15)):

$$\begin{aligned}
&[\mathcal{P}_\mu, \mathcal{Q}]_{_} = 0 \quad [\mathcal{Q}, \mathcal{Q}]_{_} = 0\\
&[\mathcal{S}_{\mu\nu}, \mathcal{Q}]_{_} = \delta_{0m}(\cdots) \quad [\mathcal{L}_{\mu\nu}, \mathcal{Q}]_{_} = \delta_{0m}(\cdots) \quad [\mathcal{M}_{\mu\nu}, \mathcal{Q}]_{_} = \delta_{0m}(\cdots).
\end{aligned} \tag{8.19}$$

[31] This is the place where the supposition $m \neq 0$ is essentially used and the proof brakes down if $m = 0$.

So, if $m \neq 0$ or if $m = 0$ and $a_3^{\pm}(\boldsymbol{k}) = a_3^{\dagger\pm}(\boldsymbol{k}) = 0$, the spin, orbital and total angular momentum operators commute with the charge operator, as it is stated by (8.11); the momentum and charge operators always commute.

9. Vacuum and normal ordering

For a general motivation regarding the introduction of the concepts of vacuum and normal ordering, the reader is referred, e.g., to [1–3] (see also [13, 14]). Below we shall concentrate on their formal aspects in an extend enough for the purposes of the present work.

Definition 9.1. The vacuum of a free vector field $\mathcal{U}$ (satisfying the Lorenz condition) is its physical state that contains no particles and has vanishing 4-momentum, (total) charge and (total) angular momentum. It is described by a state vector, denoted by $\mathcal{X}_0$ (in momentum picture) and called also the vacuum (of the field), such that:

$$\mathcal{X}_0 \neq 0 \tag{9.1a}$$

$$\mathcal{X}_0 = \tilde{\mathcal{X}}_0 \tag{9.1b}$$

$$a_s^-(\boldsymbol{k})(\mathcal{X}_0) = a_s^{\dagger -}(\boldsymbol{k})(\mathcal{X}_0) = 0 \tag{9.1c}$$

$$\langle \mathcal{X}_0 | \mathcal{X}_0 \rangle = 1 \tag{9.1d}$$

where $\langle\cdot|\cdot\rangle \colon \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ is the (Hermitian) scalar product of system's (field's) Hilbert space of states and $s = 1, 2, 3$.

It is known, this definition is in contradiction with the expressions for the dynamical variables, obtained in Sect. 6, as the latter imply infinite, instead of vanishing, characteristics for the vacuum; see (6.5)–(6.7), (6.15) and (8.2). To overcome this problem, one should redefine the dynamical variables of a free vector field via the so-called *normal ordering* of operator products (compositions) of creation and/or annihilation operators. In short, this procedure, when applied to free vector fields (satisfying the Lorenz condition), says that [1–3, 26]:

(i) The Lagrangian and the field's dynamical variables, obtained from it and containing the field operators $\mathcal{U}_\mu$ and $\mathcal{U}_\mu^\dagger$, should be written in terms of the creation and annihilation operators via (5.1)–(5.7).

(ii) Any composition (product) of creation and/or annihilation operators, possibly appearing under some integral sign(s), must be changed so that all creation operators to stand to the left of all annihilation operators.[32]

The just described procedure is known as *normal ordering (of products)* and the result of its application on some operator is called its normal form; in particular, its application on a product of creation and/or annihilation operators is called their normal product. The mapping assigning to an operator its normal form, obtained from it according to the above procedure, will be denoted by $\mathcal{N}$ and it is called *normal ordering operator* and its action on a product of creation and/or annihilation operators is defined according to the rule (ii) given above. The action of $\mathcal{N}$ on polynomials or convergent power series of creation and/or annihilation operators is extended by linearity. Evidently, the order of the creation and/or annihilation operators in some expression does not influenced the result of the action of $\mathcal{N}$ on it. The dynamical variables after normal ordering are denoted by the same symbols as before it.

It should be noticed, the normal ordering procedure, as introduced above, concerns all degrees of freedom, i.e. the ones involved in the field equations (7.4) and the operators

[32] The relative order of the creation/annihilation operators is insignificant as they commute according to (8.2).

$a_3^{\mp}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\pm}(\boldsymbol{k})|_{m=0}$, in the massless case. This simplifies temporary the consideration of massless vector fields, but does not remove the problems it contains – see Sect. 11. Moreover, in Sect. 11 arguments will be presented that the afore-given definition of normal ordering agrees with the description of electromagnetic field and that the mentioned operators should anticommute with the other ones. In principle, one can consider the normal ordering operation only for creation and annihilation operators involved in the field equations (7.4), i.e. by excluding its action on the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\pm}(\boldsymbol{k})|_{m=0}$. This will add new problems with the spin and orbital angular momentum of the vacuum as it will be, possibly, finite, but completely undetermined.

From the evident equalities

$$\begin{aligned}
\mathcal{N}\big(a_s^-(\boldsymbol{k})\circ a_t^{\dagger+}(\boldsymbol{k})\big) &= \mathcal{N}\big(a_t^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k})\big) = a_t^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k})\\
\mathcal{N}\big(a_s^{\dagger-}(\boldsymbol{k})\circ a_t^{+}(\boldsymbol{k})\big) &= \mathcal{N}\big(a_t^{+}(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k})\big) = a_t^{+}(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k})\\
\mathcal{N}\big(b^{\pm}k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}\circ b^{\mp}\big) &= \pm b^+ k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}\circ b^- \quad b^{\pm}=a_s^{\pm},a_s^{\dagger\pm}
\end{aligned}\tag{9.2}$$

and the equations (6.5)–(6.7), (6.15), and (7.18)–(7.20), we see that the dynamical variables of a free vector field (satisfying the Lorenz condition) take the following form after normal ordering:

$$\mathcal{P}_\mu = \frac{1}{1+\tau(\mathcal{U})}\sum_{s=1}^{3-\delta_{0m}}\int k_\mu|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\{a_s^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k})+a_s^+(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{9.3}$$

$$\mathcal{Q} = q\sum_{s=1}^{3-\delta_{0m}}\int\{a_s^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k})-a_s^+(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{9.4}$$

$$\begin{aligned}
\mathcal{L}_{\mu\nu} ={}& \frac{1}{1+\tau(\mathcal{U})}\sum_{s=1}^{3-\delta_{0m}}\int \mathrm{d}^3\boldsymbol{k}(x_\mu k_\nu - x_\nu k_\mu)|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\{a_s^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k})+a_s^+(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k})\}\\
&+\frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\sum_{s,s'=1}^{3}\int \mathrm{d}^3\boldsymbol{k}\, l_{\mu\nu}^{ss'}(\boldsymbol{k})\{a_s^{\dagger+}(\boldsymbol{k})\circ a_{s'}^-(\boldsymbol{k})-a_{s'}^+(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k})\}\\
&+\frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))}\sum_{s=1}^{3-\delta_{0m}}\int \mathrm{d}^3\boldsymbol{k}\Big\{a_s^{\dagger+}(\boldsymbol{k})\Big(k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}-k_\nu\overleftrightarrow{\frac{\partial}{\partial k^\mu}}\Big)\circ a_s^-(\boldsymbol{k})\\
&+a_s^+(\boldsymbol{k})\Big(k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}-k_\nu\overleftrightarrow{\frac{\partial}{\partial k^\mu}}\Big)\circ a_s^{\dagger-}(\boldsymbol{k})\Big\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}+\delta_{0m}\frac{1}{\mathrm{i}\hbar}(x^\lambda-x_0^\lambda)\mathcal{N}({}^L C_{\mu\nu\lambda})
\end{aligned}\tag{9.5}$$

$$\begin{aligned}
\mathcal{S}_{\mu\nu} ={}& \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\sum_{s,s'=1}^{3}\int \mathrm{d}^3\boldsymbol{k}\sigma_{\mu\nu}^{ss'}(\boldsymbol{k})\{a_s^{\dagger+}(\boldsymbol{k})\circ a_{s'}^-(\boldsymbol{k})-a_{s'}^+(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k})\}\\
&-\delta_{0m}\frac{1}{\mathrm{i}\hbar}(x^\lambda-x_0^\lambda)\mathcal{N}({}^S C_{\mu\nu\lambda}),
\end{aligned}\tag{9.6}$$

where $\mathcal{N}({}^L C_{\mu\nu\lambda})$ and $\mathcal{N}({}^S C_{\mu\nu\lambda})$ can easily be found by means of (9.2), (7.11) and (7.6), but we shall not need the explicit form of these operators. Similarly, the vectors of the spin (6.25) and (6.26) take the following form after normal ordering (see also (7.19)):

$$\begin{aligned}
\tilde{\mathcal{R}}_a ={}& \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\varepsilon_{abc}\int \mathrm{d}^3\boldsymbol{k}r^b(\boldsymbol{k})\{\boldsymbol{a}^{\dagger+}(\boldsymbol{k})\overset{\circ}{\times}\boldsymbol{a}^-(\boldsymbol{k})+\boldsymbol{a}^+(\boldsymbol{k})\overset{\circ}{\times}\boldsymbol{a}^{\dagger-}(\boldsymbol{k})\}^c\\
&-\delta_{0m}\frac{1}{\mathrm{i}\hbar}(x^\lambda-x_0^\lambda)\mathcal{N}({}^S C_{0a\lambda})
\end{aligned}\tag{9.7}$$

$$\boldsymbol{\mathcal{S}} = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})} \int \mathrm{d}^3\boldsymbol{k} \{ \boldsymbol{a}^{\dagger\,+}(\boldsymbol{k}) \overset{\circ}{\times} \boldsymbol{a}^{-}(\boldsymbol{k}) + \boldsymbol{a}^{+}(\boldsymbol{k}) \overset{\circ}{\times} \boldsymbol{a}^{\dagger\,-}(\boldsymbol{k}) \}^{c} - \delta_{0m}\, \hat{\boldsymbol{\mathcal{S}}} \tag{9.8}$$

with $\hat{\mathcal{S}}^a := \frac{1}{\mathrm{i}\hbar}\varepsilon^{abc}(x^\lambda - x_0^\lambda)\,\mathcal{N}({}^{S}C_{bc\lambda})$. In particular, the third component of $\boldsymbol{\mathcal{S}}$ is (cf. (6.29))

$$\begin{aligned} \mathcal{S}^3 = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})} \int \mathrm{d}^3\boldsymbol{k} \{ & a_1^{\dagger\,+}(\boldsymbol{k}) \circ a_2^{-}(\boldsymbol{k}) - a_2^{\dagger\,+}(\boldsymbol{k}) \circ a_1^{-}(\boldsymbol{k}) \\ & - a_2^{+}(\boldsymbol{k}) \circ a_1^{\dagger\,-}(\boldsymbol{k}) + a_1^{+}(\boldsymbol{k}) \circ a_2^{\dagger\,-}(\boldsymbol{k}) \}. \end{aligned} \tag{9.9}$$

The substitutions (6.30) transform the integrand in the last equality into a 'diagonal' form and preserves the ones into (9.3) and (9.4), i.e.

$$\mathcal{P}_\mu = \frac{1}{1+\tau(\mathcal{U})} \sum_{s=1}^{3-\delta_{0m}} \int k_\mu \big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{ b_s^{\dagger\,+}(\boldsymbol{k}) \circ b_s^{-}(\boldsymbol{k}) + b_s^{+}(\boldsymbol{k}) \circ b_s^{\dagger\,-}(\boldsymbol{k}) \}\, \mathrm{d}^3\boldsymbol{k} \tag{9.10}$$

$$\mathcal{Q} = q \sum_{s=1}^{3-\delta_{0m}} \int \{ b_s^{\dagger\,+}(\boldsymbol{k}) \circ b_s^{-}(\boldsymbol{k}) - a_s^{+}(\boldsymbol{k}) \circ b_s^{\dagger\,-}(\boldsymbol{k}) \}\, \mathrm{d}^3\boldsymbol{k} \tag{9.11}$$

$$\mathcal{S}^3 = \frac{\hbar}{1+\tau(\mathcal{U})} \sum_{s=1}^{2} \int (-1)^{s+1} \{ b_s^{\dagger\,+}(\boldsymbol{k}) \circ b_s^{-}(\boldsymbol{k}) - b_s^{+}(\boldsymbol{k}) \circ b_s^{\dagger\,-}(\boldsymbol{k}) \}\, \mathrm{d}^3\boldsymbol{k}. \tag{9.12}$$

From the just written expressions for the dynamical variables after normal ordering is evident that

$$\mathcal{P}_\mu(\mathcal{X}_0) = 0 \quad \mathcal{Q}(\mathcal{X}_0) = 0 \quad \mathcal{M}_{\mu\nu}(\mathcal{X}_0) = \mathcal{L}_{\mu\nu}(\mathcal{X}_0) = \mathcal{S}_{\mu\nu}(\mathcal{X}_0) = 0 \tag{9.13}$$

and, consequently, the conserved quantities of the vacuum, the 4-momentum, charge, spin and orbital angular momenta, vanish, as required by definition 9.1.

Besides the dynamical variables, the normal ordering changes the field equations (7.4) too. As the combinations quadratic in creation and annihilation, operators in the commutators in (7.4) come from the momentum operator (see (7.2)), the field equations (7.4) after normal ordering, by virtue of (9.3), read:

$$\big[a_s^{\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + a_t^{+}(\boldsymbol{q}) \circ a_t^{\dagger\,-}(\boldsymbol{q}) \big]_{\_} \pm (1+\tau(\mathcal{U})) a_s^{\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) = f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q}) \tag{9.14a}$$

$$\big[a_s^{\dagger\,\pm}(\boldsymbol{k}), a_t^{\dagger\,+}(\boldsymbol{q}) \circ a_t^{-}(\boldsymbol{q}) + a_t^{+}(\boldsymbol{q}) \circ a_t^{\dagger\,-}(\boldsymbol{q}) \big]_{\_} \pm (1+\tau(\mathcal{U})) a_s^{\dagger\,\pm}(\boldsymbol{k}) \delta_{st} \delta^3(\boldsymbol{k}-\boldsymbol{q}) = f_{st}^{\dagger\,\pm}(\boldsymbol{k},\boldsymbol{q}), \tag{9.14b}$$

$$s,t = \begin{cases} 1,2,3 & \text{for } m \neq 0 \\ 1,2 & \text{for } m = 0 \end{cases} \tag{9.14c}$$

$$\sum_{t=1}^{3-\delta_{0m}} \int q_\mu \big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q})\, \mathrm{d}^3\boldsymbol{q} = 0 \qquad \sum_{t=1}^{3-\delta_{0m}} \int q_\mu \big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}} f_{st}^{\dagger\,\pm}(\boldsymbol{k},\boldsymbol{q})\, \mathrm{d}^3\boldsymbol{q} = 0. \tag{9.14d}$$

However, one can verify that (9.14) hold identically due to the commutation relations (8.2). Once again, this demonstrates that (8.2) play a role of field equations under the suppositions made for their derivation.

As a result of (9.6), (7.19) and (8.4), we see that the commutation relations (8.5) after normal ordering transform into

$$
\begin{aligned}
[\tilde{\mathcal{S}}_{\mu\nu},\, \tilde{\mathcal{S}}_{\varkappa\lambda}]_{\_} &= [\mathcal{S}_{\mu\nu},\, \mathcal{S}_{\varkappa\lambda}]_{\_}\big|_{m\neq 0} \\
&= \frac{\hbar^2}{(1+\tau(\mathcal{U}))^2} \sum_{s,s',t=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} \{ \sigma_{\mu\nu}^{s't}(\boldsymbol{k}) \sigma_{\varkappa\lambda}^{ss'}(\boldsymbol{k}) - \sigma_{\mu\nu}^{ss'}(\boldsymbol{k}) \sigma_{\varkappa\lambda}^{s't}(\boldsymbol{k}) \} \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_t^-(\boldsymbol{k}) + a_s^+(\boldsymbol{k}) \circ a_t^{\dagger -}(\boldsymbol{k}) \\
&\qquad - \tau(\mathcal{U}) a_s^+(\boldsymbol{k}) \circ a_t^-(\boldsymbol{k}) + \tau(\mathcal{U}) a_s^{\dagger +}(\boldsymbol{k}) \circ a_t^{\dagger -}(\boldsymbol{k}) \} + \delta_{0m} f_{\mu\nu\varkappa\lambda}^{\mathcal{N}}(a_3^\pm, a_3^{\dagger\pm}), \quad (9.15)
\end{aligned}
$$

with $f_{\mu\nu\varkappa\lambda}^{\mathcal{N}}(a_3^\pm, a_3^{\dagger\pm})$ being a term whose integrand is a homogeneous expression in $a_3^\pm(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$ and which term is set equal to zero for $m \neq 0$.

The commutation relations between the spin angular momentum operator and the charge one, expressed by (8.9)–(8.11) before normal ordering, take the following form after normal ordering

$$[\tilde{\mathcal{S}}_{\mu\nu},\, \tilde{\mathcal{Q}}]_{\_} = \delta_{0m} \tilde{f}_{\mu\nu}^{\mathcal{N}}(a_3^\pm, a_3^{\dagger\pm}) \quad (9.16)$$

$$[\mathcal{S}_{\mu\nu},\, \mathcal{Q}]_{\_} = \delta_{0m} f_{\mu\nu}^{\mathcal{N}}(a_3^\pm, a_3^{\dagger\pm}) \quad (9.17)$$

$$[\mathcal{S}_{\mu\nu},\, \mathcal{Q}]_{\_}\big|_{m\neq 0} = 0. \quad (9.18)$$

These relations can be checked by means of (9.6), (9.4) and (8.4); alternatively, they are corollaries of (9.15) with $\sigma_{\varkappa\lambda}^{st}(\boldsymbol{k})$ replaced by $-\frac{1}{\mathrm{i}\hbar}(1+\tau(\mathcal{U}))q\delta^{st}$.

10. State vectors and particle interpretation

The description of the state vectors of a free vector field satisfying the Lorenz condition is practically identical with the one of a free spinor field, presented in [14]. To be obtained the former case from the latter one, the following four major changes should be made: (i) the polarization indices should run over the range 1, 2 and 3, if $m \neq 0$, or 1 and 2, if $m = 0$; (ii) the commutation relations (8.2) must replace the corresponding anticommutation ones for a free spinor field; (iii) the replacements $\sigma_{\mu\nu}^{st,\pm}(\boldsymbol{k}) \mapsto \sigma_{\mu\nu}^{st}(\boldsymbol{k})$ and $l_{\mu\nu}^{st,\pm}(\boldsymbol{k}) \mapsto l_{\mu\nu}^{st}(\boldsymbol{k})$ of the spin and orbital momentum coefficients should be made; (vi) the additional terms, depending on the polarization $s = 3$ should be taken into account, when the spin and orbital angular momentum operators of a massless field are considered. According to these changes, we present below a *mutatis mutandis* version of the corresponding considerations in [14] (see also [13].

In momentum picture, in accord with the general theory of Sect. 2, the state vectors of a vector field are spacetime-dependent, contrary to the field operators and dynamical variables constructed from them. In view of (2.12), the spacetime-dependence of a state vector $\mathcal{X}(x)$ is

$$\mathcal{X}(x) = \mathcal{U}(x, x_0)(\mathcal{X}(x_0)) \quad (10.1)$$

where x_0 is an arbitrarily fixed spacetime point and the *evolution operator* $\mathcal{U}(x, x_0)\colon \mathcal{F} \to \mathcal{F}$ is

$$\mathcal{U}(x, x_0) = \exp\Big\{ \frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu) \sum_s \int k_\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) + a_s^+(\boldsymbol{k}) \circ a_s^{\dagger -}(\boldsymbol{k}) \} \,\mathrm{d}^3\boldsymbol{k} \Big\}. \quad (10.2)$$

due to (2.2) and (9.3) (see also (2.10)–(2.12)). The operator (10.2) plays also a role of an 'S-matrix' determining the transition amplitudes between any initial and final states, say $\mathcal{X}_i(x_i)$ and $\mathcal{X}_f(x_f)$ respectively. In fact, we have

$$S_{fi}(x_f, x_i) := \langle \mathcal{X}_f(x_f) | \mathcal{X}_i(x_i) \rangle = \langle \mathcal{X}_f(x_f^{(0)}) | \mathcal{U}(x_i, x_f)(\mathcal{X}_i(x_i^{(0)})) \rangle. \quad (10.3)$$

For some purposes, the following expansion of $\mathcal{U}(x_i, x_f)$ into a power series may turn to be useful:

$$\mathcal{U}(x_i, x_f) = \mathsf{id}_{\mathcal{F}} + \sum_{n=1}^{\infty} \mathcal{U}^{(n)}(x_i, x_f) \tag{10.4}$$

$$\begin{aligned}\mathcal{U}^{(n)}(x_i, x_f) := \frac{1}{n!}(x_i^{\mu_1} - x_f^{\mu_1}) \dots (x_i^{\mu_n} - x_f^{\mu_n}) \sum_{s_1,\dots,s_n=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k}^{(1)} \dots \mathrm{d}^3\boldsymbol{k}^{(n)} k_{\mu_1}^{(1)} \cdots k_{\mu_n}^{(n)} \\ \times \big\{ a_{s_1}^{\dagger +}(\boldsymbol{k}^{(1)}) \circ a_{s_1}^{-}(\boldsymbol{k}^{(1)}) + a_{s_1}^{+}(\boldsymbol{k}^{(1)}) \circ a_{s_1}^{\dagger -}(\boldsymbol{k}^{(1)}) \big\} \\ \circ \dots \circ \big\{ a_{s_n}^{\dagger +}(\boldsymbol{k}^{(n)}) \circ a_{s_n}^{-}(\boldsymbol{k}^{(n)}) + a_{s_n}^{+}(\boldsymbol{k}^{(n)}) \circ a_{s_n}^{\dagger -}(\boldsymbol{k}^{(n)}) \big\}\end{aligned} \tag{10.5}$$

where $k_0^{(a)} = \sqrt{m^2c^2 + (\boldsymbol{k}^{(a)})^2}$, $a = 1, \dots, n$.

According to (2.14) and the considerations in Sect. 5, a state vector of a state containing n' particles and n'' antiparticles, $n', n'' \geq 0$, such that the i'^{th} particle has 4-momentum $p'_{i'}$ and polarization $s'_{i'}$ and the i''^{th} antiparticle has 4-momentum $p''_{i''}$ and polarization $s''_{i''}$, where $i' = 0, 1, \dots, n'$ and $i'' = 0, 1, \dots, n''$, is given by the equality

$$\begin{aligned}&\mathcal{X}(x; p'_1, s'_1; \dots; p'_{n'}, s'_{n'}; p''_1, s''_1; \dots; p''_{n''}, s''_{n''}) \\ &\quad = \frac{1}{\sqrt{n'!n''!}} \exp\Big\{ \frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu) \sum_{i'=1}^{n'} (p'_{i'})_\mu + \frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu) \sum_{i''=1}^{n''} (p''_{i''})_\mu \Big\} \\ &\qquad \times \big(a_{s'_1}^{+}(\boldsymbol{p}'_1) \circ \dots \circ a_{s'_{n'}}^{+}(\boldsymbol{p}'_{n'}) \circ a_{s''_1}^{\dagger +}(\boldsymbol{p}''_1) \circ \dots \circ a_{s''_{n''}}^{\dagger +}(\boldsymbol{p}''_{n''}) \big)(\mathcal{X}_0),\end{aligned} \tag{10.6}$$

where, in view of the commutation relations (8.2), the order of the creation operators is inessential. If $n' = 0$ (resp. $n'' = 0$), the particle (resp. antiparticle) creation operators and the first (resp. second) sum in the exponent should be absent. In particular, the vacuum corresponds to (10.6) with $n' = n'' = 0$. The state vector (10.6) is an eigenvector of the momentum operator (9.3) with eigenvalue (4-momentum) $\sum_{i'=1}^{n'} p'_{i'} + \sum_{i''=1}^{n''} p''_{i''}$ and is also an eigenvector of the charge operator (9.4) with eigenvalue $(-q)(n' - n'')$.[33]

The reader may verify, using (8.2) and (5.19), that the transition amplitude between two states of a vector field, like (10.6), is:

$$\begin{aligned}&\langle\, \mathcal{X}(y; q'_1, t'_1; \dots; q'_{n'}, t'_{n'}; q''_1, t''_1; \dots; q''_{n''}, t''_{n''}) \\ &\qquad\qquad | \, \mathcal{X}(x; p'_1, s'_1; \dots; p'_{m'}, s'_{m'}; p''_1, s''_1; \dots; p''_{m''}, s''_{m''}) \rangle \\ &= \frac{1}{n'!n''!} \delta_{m'n'} \delta_{m''n''} \exp\Big\{ \frac{1}{\mathrm{i}\hbar}(x^\mu - y^\mu) \sum_{i'=1}^{n'} (p'_{i'})_\mu + \frac{1}{\mathrm{i}\hbar}(x^\mu - y^\mu) \sum_{i''=1}^{n''} (p''_{i''})_\mu \Big\} \\ &\quad \times \sum_{(i'_1,\dots,i'_{n'})} \delta_{s'_{n'} t'_{i'_1}} \delta^3(\boldsymbol{p}'_{n'} - \boldsymbol{q}'_{i'_1}) \delta_{s'_{n'-1} t'_{i'_2}} \delta^3(\boldsymbol{p}'_{n'-1} - \boldsymbol{q}'_{i'_2}) \dots \delta_{s'_1 t'_{i'_{n'}}} \delta^3(\boldsymbol{p}'_1 - \boldsymbol{q}'_{i'_{n'}}) \\ &\quad \times \sum_{(i''_1,\dots,i''_{n''})} \delta_{s''_{n''} t''_{i''_1}} \delta^3(\boldsymbol{p}''_{n''} - \boldsymbol{q}''_{i''_1}) \delta_{s''_{n''-1} t''_{i''_2}} \delta^3(\boldsymbol{p}''_{n''-1} - \boldsymbol{q}''_{i''_2}) \dots \delta_{s''_1 t''_{i''_{n''}}} \delta^3(\boldsymbol{p}''_1 - \boldsymbol{q}''_{i''_{n''}})\end{aligned} \tag{10.7}$$

where the summations are over all permutations $(i'_1, \dots, i'_{n'})$ of $(1, \dots, n')$ and $(i''_1, \dots, i''_{n''})$ of $(1, \dots, n'')$. The conclusions from this formula are similar to the ones concerning free scalar or spinor fields [13, 14]. For instance, the only non-forbidden transition from an n'-particle + n''-antiparticle state is into n'-particle + n''-antiparticle state; the both states may

[33] Recall (see Sect. 5), the operator $a_s^{+}(\boldsymbol{k})$ creates a particle with 4-momentum k_μ and charge $-q$, while $a_s^{\dagger +}(\boldsymbol{k})$ creates a particle with 4-momentum k_μ and charge $+q$, where, in the both cases, $k_0 = \sqrt{m^2c^2 + \boldsymbol{k}^2}$. See also equations (10.9)–(10.12) below.

differ only in the spacetime positions of the (anti)particles in them. This result is quite natural as we are dealing with free particles/fields.

In particular, if $\mathcal{X}_n$ denotes any state containing n particles and/or antiparticles, $n = 0, 1, \dots$, then (10.7) says that

$$\langle \mathcal{X}_n | \mathcal{X}_0 \rangle = \delta_{n0}, \tag{10.8}$$

which expresses the stability of the vacuum.

Consider the one (anti)particle states $a_t^+(\boldsymbol{p})(\mathcal{X}_0)$ and $a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)$, with $t = 1, 2, 3$, if $m \neq 0$, or $t = 1, 2$, if $m = 0$. Applying (9.3)–(9.10) and (8.2), we find ($p_0 := \sqrt{m^2c^2 + \boldsymbol{p}^2}$):[34]

$$\begin{aligned} \mathcal{P}_\mu\big(a_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) &= p_\mu a_t^+(\boldsymbol{p})(\mathcal{X}_0) \qquad \mathcal{Q}\big(a_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) = -q a_t^+(\boldsymbol{p})(\mathcal{X}_0) \\ \mathcal{P}_\mu\big(a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) &= p_\mu a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0) \quad \mathcal{Q}\big(a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) = +q a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0) \end{aligned} \tag{10.9}$$

$$\begin{aligned} \mathcal{S}_{\mu\nu}\big(a_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) &= -\mathrm{i}\hbar \sum_{s=1}^{3-\delta_{0m}} \sigma_{\mu\nu}^{ts}(\boldsymbol{p}) a_s^+(\boldsymbol{p})(\mathcal{X}_0) - \delta_{0m}(\cdots)(\mathcal{X}_0) \\ \mathcal{S}_{\mu\nu}\big(a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) &= -\mathrm{i}\hbar \sum_{s=1}^{3-\delta_{0m}} \sigma_{\mu\nu}^{ts}(\boldsymbol{p}) a_s^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0) - \delta_{0m}(\cdots)_\dagger(\mathcal{X}_0) \end{aligned} \tag{10.10}$$

$$\begin{aligned} \mathcal{S}^3\big(a_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) &= \mathrm{i}\hbar\{\delta_{t2}a_1^+(\boldsymbol{p}) - \delta_{t1}a_2^+(\boldsymbol{p})\}(\mathcal{X}_0) \\ \mathcal{S}^3\big(a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) &= \mathrm{i}\hbar\{\delta_{t2}a_1^{\dagger +}(\boldsymbol{p}) - \delta_{t1}a_2^{\dagger +}(\boldsymbol{p})\}(\mathcal{X}_0) \end{aligned} \tag{10.11}$$

$$\begin{aligned} \mathcal{L}_{\mu\nu}(x)\big(a_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) = &\Big\{(x_\mu p_\nu - x_\nu p_\mu) - \mathrm{i}\hbar\Big(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}\Big)\Big\}\big(a_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) \\ &- \mathrm{i}\hbar \sum_{s=1}^{3-\delta_{0m}} l_{\mu\nu}^{ts}(\boldsymbol{p})\big(a_s^+(\boldsymbol{p})(\mathcal{X}_0)\big) - \delta_{0m}(\cdots)(\mathcal{X}_0) \\ \mathcal{L}_{\mu\nu}(x)\big(a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) = &\Big\{(x_\mu p_\nu - x_\nu p_\mu) - \mathrm{i}\hbar\Big(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}\Big)\Big\}\big(a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) \\ &- \mathrm{i}\hbar \sum_{s=1}^{3-\delta_{0m}} l_{\mu\nu}^{ts}(\boldsymbol{p})\big(a_s^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) - \delta_{0m}(\cdots)_\dagger(\mathcal{X}_0) \end{aligned} \tag{10.12}$$

where $q\tau(\mathcal{U}) \equiv 0$ was used and $(\cdots)$ and $(\cdots)_\dagger$ denote expressions whose integrands are homogeneous with respect to $a_3^\pm(\boldsymbol{p})$ and $a_3^{\dagger\pm}(\boldsymbol{p})$. It should be remarked the agreement of (10.9)–(10.12) with (5.20).[35]

If one uses the operators $b_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ instead of $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ (see (6.30)), the equations (10.9) and (10.11) will read ($p_0 := \sqrt{m^2c^2 + \boldsymbol{p}^2}$):

$$\begin{aligned} \mathcal{P}_\mu\big(b_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) &= p_\mu b_t^+(\boldsymbol{p})(\mathcal{X}_0) \qquad \mathcal{Q}\big(b_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) = -q b_t^+(\boldsymbol{p})(\mathcal{X}_0) \\ \mathcal{P}_\mu\big(b_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) &= p_\mu b_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0) \quad \mathcal{Q}\big(b_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) = +q b_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0) \end{aligned} \tag{10.13}$$

$$\begin{aligned} \mathcal{S}^3\big(b_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) &= \hbar\{\delta_{t1}b_1^+(\boldsymbol{p}) - \delta_{t2}b_2^+(\boldsymbol{p})\}(\mathcal{X}_0) \\ \mathcal{S}^3\big(b_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) &= \hbar\{\delta_{t1}b_1^{\dagger +}(\boldsymbol{p}) - \delta_{t2}b_2^{\dagger +}(\boldsymbol{p})\}(\mathcal{X}_0) \end{aligned} \tag{10.14}$$

We should emphasize, the equations (10.9)–(10.14) do *not* concern the massless case, $m = 0$, with polarization $t = 3$, i.e. the (state?) vectors $a_3^+(\boldsymbol{p})\big|_{m=0}(\mathcal{X}_0)$ and $a_3^{\dagger +}(\boldsymbol{p})\big|_{m=0}(\mathcal{X}_0)$ have undetermined 4-momentum, charge, spin and orbital angular momentum. This is quite

[34] The easiest way to derive (10.12) is by applying (2.24), (7.2), (7.3) and (8.2). Notice, in Heisenberg picture and in terms of the Heisenberg creation/annihilation operators (6.36), equations (10.12) read $\tilde{\mathcal{L}}_{\mu\nu}\big(a_t^+(\boldsymbol{p})(\mathcal{X}_0)\big) = 0$ and $\tilde{\mathcal{L}}_{\mu\nu}\big(a_t^{\dagger +}(\boldsymbol{p})(\mathcal{X}_0)\big) = 0$ which is quite understandable in view of the fact that $\tilde{\mathcal{L}}_{\mu\nu}$ is, in a sense, the average orbital momentum with respect to all spacetime points, while $\mathcal{L}_{\mu\nu}(x, x_0)$ is the one relative to x and x_0; the dependence on x_0 being hidden in the $\mathcal{L}_{\mu\nu}$, $a_t^+(\boldsymbol{p})$ and $a_t^{\dagger +}(\boldsymbol{p})$.

[35] If the r.h.s. of (2.33) is with an opposite sign, this agreement will be lost.

understandable as the operators $a_3^{\pm}(\boldsymbol{p})\big|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{p})\big|_{m=0}$ do not enter in the field equations (7.4) and, consequently, remain completely arbitrary. (For more details on this situation, see Sect. 11.)

On the formulae (10.9)–(10.14) is based the particle interpretation of quantum field theory (in Lagrangian formalism) of free vector field satisfying the Lorenz condition. According to them, the state vectors produced by the operators $a_s^{+}(\boldsymbol{p})$ and $b_s^{+}(\boldsymbol{p})$ from the vacuum $\mathcal{X}_0$ can be interpreted as ones representing particles with 4-momentum $(\sqrt{m^2c^2+\boldsymbol{p}^2},\boldsymbol{p})$ and charge $(-q)$; the spin and orbital angular momentum of these vectors is not definite. The states $a_s^{+}(\boldsymbol{p})(\mathcal{X}_0)$ do not have a definite projection of the vector of spin on the direction of movement, i.e. along $\boldsymbol{p}$ (for $\boldsymbol{p}\neq\boldsymbol{0}$), but, if $m\neq 0$, the one of $b_s^{+}(\boldsymbol{p})(\mathcal{X}_0)$ is equal to $(-1)^{s+1}\hbar$, if $s=1,2$, or to zero, if $s=3$. Similarly, the state vectors $a_s^{\dagger\,+}(\boldsymbol{p})(\mathcal{X}_0)$ and $b_s^{\dagger\,+}(\boldsymbol{p})(\mathcal{X}_0)$ should be interpreted as ones representing particles, called *antiparticles* (with respect to the ones created by $a_s^{+}(\boldsymbol{p})$ and $b_s^{+}(\boldsymbol{p})$), with the same characteristics but the charge, which for them is equal to $(+q)$. For this reason, the particles and antiparticles of a neutral (Hermitian), $q=0$, field coincide.

Notice, the above interpretation of the creation and annihilation operators does not concern these operators for a massless field and polarization along the 3-momentum, i.e. for $(m,s)=(0,3)$. Besides, in the massless case, the just said about the spin projection of the states $b_s^{+}(\boldsymbol{p})(\mathcal{X}_0)$ and $b_s^{\dagger\,+}(\boldsymbol{p})(\mathcal{X}_0)$) is not valid unless the last terms, proportional to δ_{0m}, in (10.10) vanish.

11. The massless case. Electromagnetic field in Lorenz gauge

A simple overview of the preceding sections reveals that the zero-mass case, $m=0$, is more or less an exception of the general considerations. The cause for this is that the Lorenz condition, expressed by (3.18), is external to the Lagrangian formalism for a massless free vector field, contrary to the massive case, $m\neq 0$. However, this condition does not contradict to the formalism and, as we demonstrated, it can be developed to a reasonable extend.

Practically, the only problem in the massless case, we have met, is with the physical meaning/interpretation of the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$.[36] The first indications for it were the last two equations in (5.20c) (valid if $m=0$), which were derived from the external to the Lagrangian formalism equation (2.44) and, hence, can be neglected if one follows rigorously the Lagrangian field theory; moreover, in Sect. 6, we proved that the equations (6.18) imply $a_s^{\pm}(\boldsymbol{k}) = a_s^{\dagger\,\pm}(\boldsymbol{k}) = 0$ for $s=1,2$, which, in view of the further development of the theory, is unacceptable. The really serious problem with the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ is that they have vanishing contribution to the momentum operator (6.5) and charge operator (6.6), but they have, generally, non-vanishing one to the orbital and spin angular momentum operators (6.7) and (6.15), respectively.[37] (See also the vectors of spin (6.25) and (6.26) in which these operators enter via (6.27).) In connection with the (possible) interpretation in terms of particles, this means that $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ describe creation/annihilation of neutral massless particles with vanishing 4-momentum and charge, but, generally, non-zero spin and orbital angular momentum. Besides, the last two characteristics of the (hypothetical) particles with state vectors $a_3^{+}(\boldsymbol{k})|_{m=0}(\mathcal{X}_0)$ and $a_3^{\dagger\,+}(\boldsymbol{k})|_{m=0}(\mathcal{X}_0)$ can be completely arbitrary since the field equation (7.4) do not impose on the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ any restrictions.[38]

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[36] From pure mathematical viewpoint, everything is in order and no problems arise.

[37] To save some space, here our considerations do not take into account the normal ordering, i.e. they concern the theory before it; *vide infra*.

[38] However, the third component of the vector of spin $\boldsymbol{\mathcal{S}}$ does not depend on $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$

The above discussion leads to the following conclusion. The Lagrangian formalism, without further assumptions/hypotheses, cannot give any information about the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\pm}(\boldsymbol{k})|_{m=0}$ and, consequently, leaves them as free parameters of the quantum field theory of massless free vector field satisfying the Lorenz condition and described by the Lagrangian (3.7). Thus, these operators are carries of a completely arbitrary degrees of freedom, which have a non-vanishing contribution to the spin and orbital angular momentum operators (6.15) and (6.7), respectively, unless before normal ordering we have

$$\sum_{s=1,2}\int \mathrm{d}^3\boldsymbol{k}\sigma_{\mu\nu}^{s3}(\boldsymbol{k})\big|_{m=0}\{a_s^{\dagger+}(\boldsymbol{k})\circ a_3^-(\boldsymbol{k}) - a_s^{\dagger-}(\boldsymbol{k})\circ a_3^+(\boldsymbol{k}) \\ - a_3^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k}) + a_3^{\dagger-}(\boldsymbol{k})\circ a_s^+(\boldsymbol{k})\}\big|_{m=0} = 0 \quad (11.1a)$$

$$\sum_{s=1,2}\int \mathrm{d}^3\boldsymbol{k}l_{\mu\nu}^{s3}(\boldsymbol{k})\big|_{m=0}\{a_s^{\dagger+}(\boldsymbol{k})\circ a_3^-(\boldsymbol{k}) - a_s^{\dagger-}(\boldsymbol{k})\circ a_3^+(\boldsymbol{k}) \\ - a_3^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k}) + a_3^{\dagger-}(\boldsymbol{k})\circ a_s^+(\boldsymbol{k})\}\big|_{m=0} = 0, \quad (11.1b)$$

where we have used the skewsymmetry of the quantities (6.8) and (6.16). The just-presented considerations concern the theory before (second) quantization, i.e. before imposing the commutation relations (8.2), and normal ordering. However, since the quantization procedure does not concern the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\pm}(\boldsymbol{k})|_{m=0}$ (see Sect. 8), the above-said remains completely valid after these procedures, provided one takes into account the expressions (9.3)–(9.8) for the dynamical variables after normal ordering. In particular, after normal ordering, the spin and orbital angular momentum operators (9.6) and (9.5), respectively, will be independent of $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\pm}(\boldsymbol{k})|_{m=0}$ iff

$$\sum_{s=1,2}\int \mathrm{d}^3\boldsymbol{k}\sigma_{\mu\nu}^{s3}(\boldsymbol{k})\big|_{m=0}\{a_s^{\dagger+}(\boldsymbol{k})\circ a_3^-(\boldsymbol{k}) - a_3^+(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k}) \\ - a_3^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k}) + a_s^+(\boldsymbol{k})\circ a_3^{\dagger-}(\boldsymbol{k})\}\big|_{m=0} = 0 \quad (11.2a)$$

$$\sum_{s=1,2}\int \mathrm{d}^3\boldsymbol{k}l_{\mu\nu}^{s3}(\boldsymbol{k})\big|_{m=0}\{a_s^{\dagger+}(\boldsymbol{k})\circ a_3^-(\boldsymbol{k}) - a_3^+(\boldsymbol{k})\circ a_s^{\dagger-}(\boldsymbol{k}) \\ - a_3^{\dagger+}(\boldsymbol{k})\circ a_s^-(\boldsymbol{k}) + a_s^+(\boldsymbol{k})\circ a_3^{\dagger-}(\boldsymbol{k})\}\big|_{m=0} = 0. \quad (11.2b)$$

Thus, if one wants to construct a sensible *physical* theory of a massless free vector field satisfying the Lorenz condition, new assumptions to the Lagrangian formalism should be added. At this point, there is a room for different kinds of speculations. Here are two such possibilities.

One can demand, as an additional condition, the fulfillment of the field equations (7.4) for $m = 0$ and any polarization indices, i.e. for $s, t = 1, 2, 3$, instead only for $s, t = 1, 2$ obtained from the Lagrangian formalism. This will entail the vanishment of all of the quantities $B_{ss'}^{\mu,\mp}(\boldsymbol{q})$, i.e. (7.6) will be replaced with

$$B_{ss'}^{\mu,\mp}(\boldsymbol{q}) \equiv 0 \qquad s, s' = 1, 2, 3 \quad (11.3)$$

which, in its turn, leads to (see (7.11))

$${}^{S}C_{\mu\nu}^{\lambda} = {}^{L}C_{\mu\nu}^{\lambda} = 0 \quad (11.4)$$

— see (6.29) and (10.11).

and, consequently, to the commutativity of the spin angular momentum and momentum operators, etc. (see (7.9)–(7.20)). Other consequence of the above assumption will be the validity of the commutation relations (8.2) for arbitrary polarization indices $s, t = 1, 2, 3$ in the massless case. As a result of them and the normal ordering procedure, the vectors $a_3^{+}(\boldsymbol{k})|_{m=0}(\mathcal{X}_0)$ and $a_3^{\dagger +}(\boldsymbol{k})|_{m=0}(\mathcal{X}_0)$ will describe states with vanishing 4-momentum and charge and, generally, non-vanishing spin and orbital angular momentum. It seems, states/particles with such characteristics have not been observed until now. This state of affairs can be improved by adding to the integrands in (9.3) and (9.4) terms proportional to $a_3^{\dagger +}(\boldsymbol{k}) \circ a_3^{-}(\boldsymbol{k})$ and $a_3^{+}(\boldsymbol{k}) \circ a_3^{\dagger -}(\boldsymbol{k})$ in the massless case, but such a game with adjustment of theory's parameters is out of the scope of the present work.

The second possible solution of the problem(s) with the zero-mass case, we would like to explore, does not require drastical changes of the formalism as the preceding one. In it to the Lagrangian formalism are added, as subsidiary conditions, the equations (11.2) or (11.1), depending if the normal ordering is or is not taken into account, respectively. In this way only the operators of spin and orbital angular momentum are changed, viz. before normal ordering they read (see (6.15), (6.7), (7.19), (7.20) and notice that the above assumption entails (11.4))

$$\mathcal{S}_{\mu\nu} = \tilde{\mathcal{S}}_{\mu\nu} = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})} \sum_{s,s'=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} \sigma_{\mu\nu}^{ss'}(\boldsymbol{k}) \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_{s'}^{-}(\boldsymbol{k}) - a_s^{\dagger -}(\boldsymbol{k}) \circ a_{s'}^{+}(\boldsymbol{k}) \} \tag{11.5}$$

$$\begin{aligned} \mathcal{L}_{\mu\nu} &- (x_\mu - x_{0\,\mu}) \mathcal{P}_\nu + (x_\nu - x_{0\,\nu}) \mathcal{P}_\mu = \tilde{\mathcal{L}}_{\mu\nu} \\ &= \frac{1}{1+\tau(\mathcal{U})} \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} (x_{0\,\mu} k_\nu - x_{0\,\nu} k_\mu)|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_s^{-}(\boldsymbol{k}) + a_s^{\dagger -}(\boldsymbol{k}) \circ a_s^{+}(\boldsymbol{k}) \} \\ &\quad + \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})} \sum_{s,s'=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k}\, l_{\mu\nu}^{ss'}(\boldsymbol{k}) \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_{s'}^{-}(\boldsymbol{k}) - a_s^{\dagger -}(\boldsymbol{k}) \circ a_{s'}^{+}(\boldsymbol{k}) \} \\ &\quad + \frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))} \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a_s^{\dagger +}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{-}(\boldsymbol{k}) \\ &\qquad - a_s^{\dagger -}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{+}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \end{aligned} \tag{11.6}$$

and, after normal ordering, they take the form (see (9.6) and (9.5))

$$\mathcal{S}_{\mu\nu} = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})} \sum_{s,s'=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} \sigma_{\mu\nu}^{ss'}(\boldsymbol{k}) \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_{s'}^{-}(\boldsymbol{k}) - a_{s'}^{+}(\boldsymbol{k}) \circ a_s^{\dagger -}(\boldsymbol{k}) \} \tag{11.7}$$

$$\begin{aligned} \mathcal{L}_{\mu\nu} &= \frac{1}{1+\tau(\mathcal{U})} \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} (x_\mu k_\nu - x_\nu k_\mu)|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}} \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_s^{-}(\boldsymbol{k}) + a_s^{+}(\boldsymbol{k}) \circ a_s^{\dagger -}(\boldsymbol{k}) \} \\ &\quad + \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})} \sum_{s,s'=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k}\, l_{\mu\nu}^{ss'}(\boldsymbol{k}) \{ a_s^{\dagger +}(\boldsymbol{k}) \circ a_{s'}^{-}(\boldsymbol{k}) - a_{s'}^{+}(\boldsymbol{k}) \circ a_s^{\dagger -}(\boldsymbol{k}) \} \\ &\quad + \frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))} \sum_{s=1}^{3-\delta_{0m}} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a_s^{\dagger +}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{-}(\boldsymbol{k}) \\ &\qquad + a_s^{+}(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^{\dagger -}(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}. \end{aligned} \tag{11.8}$$

So, formally, the replacement $\sum_{s,s'=1}^{3} \mapsto \sum_{s,s'=1}^{3-\delta_{0m}}$ should be made and the terms proportional to δ_{0m} should be deleted. As a result of these changes, all terms proportional to δ_{0m} in all equations, starting from (7.9) onwards, will disappear, i.e., for any m, we have

$$\delta_{0m} \times (\cdots) = 0, \tag{11.9}$$

where the dots stand for some expressions, which depend on $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ for $m = 0$ and are set to zero for $m \neq 0$; in particular, the equations (11.4) hold (see (7.11) and (7.5)), but (11.3) do not.

In this way, the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ disappear from all dynamical variables. So, if we extend the particle interpretation on them,[39] these operators will describe creation/annihilation of massless particles with vanishing 4-momentum, charge, spin and orbital angular momentum. Naturally, such 'particles' are completely unobservable. Thus, the properties of the states $a_3^{+}(\boldsymbol{k})|_{m=0}(\mathcal{X}_0)$ and $a_3^{\dagger\,+}(\boldsymbol{k})|_{m=0}(\mathcal{X}_0)$ are similar to the ones of the vacuum (see (9.13)), but their identification with the vacuum $\mathcal{X}_0$ requires additional and, in a sense, artificial hypotheses for a self-consistent development of the theory.[40]

So, assuming the validity of (11.1) or (11.2), we see that these equations are the only place in the theory, where the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ essentially appear.[41] In fact, these equations should be regarded as equations of motion for the mentioned operators, which operators do not enter in the field equations (7.4) or in the commutation relations (8.2). The equations (11.1) or (11.2) possess always the trivial solution

$$a_3^{\pm}(\boldsymbol{k})|_{m=0} = 0 \quad a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0} = 0, \tag{11.10}$$

which agrees with the definition 9.1 of the vacuum, but they may have and other solutions. Since, at the moment, it seems that the operators $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ cannot lead to some physically measurable results, we shall not investigate the problem for existence of solutions of (11.1) or (11.2), different from (11.10).

Regardless of the fact that equations (11.1) or (11.2) exclude a contribution of $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ from the dynamical variables, the initial operators $\mathcal{U}_\mu$ and $\mathcal{U}_\mu^{\dagger}$ depend on them via the operators $\mathcal{U}_\mu^{\pm}(\boldsymbol{k})$ and $\mathcal{U}_\mu^{\dagger\,\pm}(\boldsymbol{k})$ – see (5.18) and (5.6)–(5.7). As a consequence of this, the commutation relations between different combinations of $\mathcal{U}_\mu^{\pm}(\boldsymbol{k})$ and $\mathcal{U}_\mu^{\dagger\,\pm}(\boldsymbol{k})$ also depend on $a_3^{\pm}(\boldsymbol{k})|_{m=0}$ and $a_3^{\dagger\,\pm}(\boldsymbol{k})|_{m=0}$ in the massless case. Moreover, one cannot calculate these relations without additional assumptions, like (11.10) or the validity of (8.2) for any $s, t = 1, 2, 3$ in the massless case. Besides, the result depends essentially on the additional condition(s) one assumes in the massless case. For instance, if we assume (8.2) to hold for any $s, t = 1, 2, 3$, when $m = 0$, then, for $(\boldsymbol{k}, m) \neq (\boldsymbol{0}, 0)$, we get

$$[\mathcal{U}_\mu^{\varepsilon}(\boldsymbol{k}), \mathcal{U}_\mu^{\varepsilon'}(\boldsymbol{k}')]_{\_} = f(\varepsilon, \varepsilon')\big\{2c(2\pi\hbar)^3\sqrt{m^2c^2 + \boldsymbol{k}^2}\big\}^{-1}\delta^3(\boldsymbol{k} - \boldsymbol{k}')\sum_{s=1}^{3}\{v_\mu^s(\boldsymbol{k})v_\nu^s(\boldsymbol{k})\}, \tag{11.11}$$

where $\varepsilon, \varepsilon' = \pm, \mp, \dagger\pm, \dagger\mp$ and we have applied (5.17) and (8.2) in the form

$$[a_s^{\varepsilon}(\boldsymbol{k}), a_t^{\varepsilon'}(\boldsymbol{k}')]_{\_} = f(\varepsilon, \varepsilon')\delta_{st}\delta^3(\boldsymbol{k} - \boldsymbol{k}') \tag{11.12a}$$

$$f(\varepsilon, \varepsilon') := \begin{cases} 0 & \text{for } \varepsilon, \varepsilon' = \pm, \dagger\pm \\ \pm\tau(U) & \text{for } (\varepsilon, \varepsilon') = (\mp, \pm), (\dagger\mp, \dagger\pm) \\ \pm 1 & \text{for } (\varepsilon, \varepsilon') = (\mp, \dagger\pm), (\dagger\mp, \pm) \end{cases} . \tag{11.12b}$$

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[39] Such an extension requires the fulfillment of the commutation relations (8.2) for $s = 3$ and/or $t = 3$ in the massless case.

[40] For instance, in a case of a neutral field, when $a_s^{\dagger\,\pm}(\boldsymbol{k}) = a_s^{\pm}(\boldsymbol{k})$, one can satisfy (11.1) by requiring $[a_3^{+}(\boldsymbol{k}), a_s^{-}(\boldsymbol{k})]_{+}|_{m=0} = [a_3^{-}(\boldsymbol{k}), a_s^{+}(\boldsymbol{k})]_{+}|_{m=0}$, with $s = 1, 2$ and $[A, B]_{+} := A \circ B + B \circ A$; in particular, this will be valid if we assume $[a_3^{\pm}(\boldsymbol{k}), a_s^{\mp}(\boldsymbol{k})]_{+}|_{m=0} = 0$ for $s = 1, 2$.

[41] The initial operators $\mathcal{U}_\mu$ and $\mathcal{U}_\mu^{\dagger}$ depend on these operators too — *vide infra*.

Combining (11.11) and (4.26), we obtain $((\boldsymbol{k},m)\neq(\boldsymbol{0},0))$

$$[\mathcal{U}_\mu^\varepsilon(\boldsymbol{k}),\mathcal{U}_\mu^{\varepsilon'}(\boldsymbol{k}')]_{\_}\big|_{m\neq0} = f(\varepsilon,\varepsilon')\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\}^{-1}\delta^3(\boldsymbol{k}-\boldsymbol{k}') \times\begin{cases}-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{m^2c^2} & \text{for } \boldsymbol{k}\neq\boldsymbol{0}\\ \delta_{\mu\nu} & \text{for } \boldsymbol{k}=\boldsymbol{0} \text{ and } \mu,\nu=1,2,3\\ 0 & \text{otherwise}\end{cases} \tag{11.13a}$$

$$[\mathcal{U}_\mu^\varepsilon(\boldsymbol{k}),\mathcal{U}_\mu^{\varepsilon'}(\boldsymbol{k}')]_{\_}\big|_{m=0} = f(\varepsilon,\varepsilon')\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\}^{-1}\delta^3(\boldsymbol{k}-\boldsymbol{k}') \times\Big(-\eta_{\mu\nu}+\frac{k_\mu k_\nu}{\boldsymbol{k}^2}\times\begin{cases}2 & \text{for } \mu=\nu=0\\ 0 & \text{for } \mu,\nu=1,2,3\\ 1 & \text{otherwise}\end{cases}\Big). \tag{11.13b}$$

On the other hand, if we assume (11.10), then (11.11) must be replaced with

$$[\mathcal{U}_\mu^\varepsilon(\boldsymbol{k}),\mathcal{U}_\mu^{\varepsilon'}(\boldsymbol{k}')]_{\_} = f(\varepsilon,\varepsilon')\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\}^{-1}\delta^3(\boldsymbol{k}-\boldsymbol{k}')\sum_{s=1}^{3-\delta_{0m}}\{v_\mu^s(\boldsymbol{k})v_\nu^s(\boldsymbol{k})\}, \tag{11.14}$$

as a consequence of (5.17), (11.12) with $s,t=\begin{cases}1,2,3 & \text{if } m\neq0\\ 1,2 & \text{if } m=0\end{cases}$, and (11.10). So, for $m\neq0$, the relation (11.14) reduces to (11.13a) (see (4.26)),[42] but for $m=0$ it reads $((\boldsymbol{k},m)\neq(\boldsymbol{0},0))$

$$[\mathcal{U}_\mu^\varepsilon(\boldsymbol{k}),\mathcal{U}_\mu^{\varepsilon'}(\boldsymbol{k}')]_{\_}\big|_{m=0} = f(\varepsilon,\varepsilon')\{2c(2\pi\hbar)^3\sqrt{m^2c^2+\boldsymbol{k}^2}\}^{-1}\delta^3(\boldsymbol{k}-\boldsymbol{k}')\times\begin{cases}\delta_{\mu\nu} & \text{for } \mu,\nu=1,2\\ 0 & \text{otherwise}\end{cases}, \tag{11.15}$$

due to (4.27). Evidently, the equations (11.13a) and (11.15) coincide for $\mu,\nu=1,2$ but otherwise are, generally, different.

It is now time to be paid special attention to the *electromagnetic field*.[43] As it is well known [1, 2, 4, 16], this field is a massless neutral vector field whose operators, called the electromagnetic potentials, are usually denoted by $\mathcal{A}_\mu$ and are such that

$$\mathcal{A}_\mu^\dagger=\mathcal{A}_\mu. \tag{11.16}$$

The (second) quantization of electromagnetic field meets some difficulties, described in *loc. cit.*, the causes for which are well-described in [15, § 82] (see also [6]). The closest to our approach is the so-called Gupta-Bleuler quantization [1, 6, 15, 16] in the way it is described in [1]. However, our method is quite different from it as we quantize only the independent degrees of freedom, as a result of which there is no need of considering indefinite metric, 'time' ('scalar') photons and similar objects.[44] The idea of most such methods is to be started from some Lagrangian, to be applied the standard canonical quantization procedure [3, 15], and, then, to the electromagnetic potentials to be imposed some subsidiary conditions, called *gauge conditions*, by means of which is (partially) fixed the freedom in the field operators left by the field equations.

In our scheme, the free electromagnetic field is described via 4 Hermitian operators $\mathcal{A}_\mu$ (for which $\tau(A)=1$ — see (3.2)), the Lagrangian (3.7) with $m=0$, i.e.

$$\mathcal{L}=\frac{1}{2}c^2[\mathcal{A}_\nu,\mathcal{P}_\mu]_{\_}\circ[\mathcal{A}^\nu,\mathcal{P}^\mu]_{\_}-\frac{1}{2}c^2[\mathcal{A}^\mu,\mathcal{P}_\mu]_{\_}\circ[\mathcal{A}^\nu,\mathcal{P}_\nu]_{\_} \tag{11.17}$$

and the Lorenz conditions (3.18) with $\mathcal{U}=\mathcal{A}$, i.e.

$$[\mathcal{A}_\mu,\mathcal{P}^\mu]_{\_}=0. \tag{11.18}$$

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[42] For $m\neq0$ and $\boldsymbol{k}\neq\boldsymbol{0}$, as one can expect, the commutation relations (8.2) and (11.13a) reproduce, due to (6.36), the known ones for a massive free vector field in Heisenberg picture [1, 2, 16].

[43] In fact, the description of electromagnetic field was the primary main reason for the inclusion of the massless case in the considerations in the preceding sections.

[44] The only such a problem we meet is connected with the 'longitudinal' photons — *vide infra.*

It should be emphasized, the Lorenz condition (11.18) is imposed directly on the field operators, not on the 'physical' states etc. as in the Gupta-Bleuler formalism. So, (11.16)–(11.18) describe an electromagnetic field in *Lorenz gauge*.

Thus, to specialize the general theory form the preceding sections to the case of electromagnetic field, one should put in it (see (11.16) and sections 5 and 6)

$$m = 0 \quad \mathcal{U} = \mathcal{A} \quad \tau(\mathcal{U}) = 1 \quad \mathcal{A}_\mu^\dagger = \mathcal{A}_\mu \quad \mathcal{A}_\mu^{\dagger\pm} = \mathcal{A}_\mu^\pm \quad \mathcal{A}_\mu^{\dagger\pm}(\boldsymbol{k}) = \mathcal{A}_\mu^\pm(\boldsymbol{k}) \quad a_s^{\dagger\pm}(\boldsymbol{k}) = a_s^\pm(\boldsymbol{k}). \tag{11.19}$$

It is important to be emphasized, the equations (11.19) reduce (11.1) to

$$\sum_{s=1,2} \int \mathrm{d}^3\boldsymbol{k} \sigma_{\mu\nu}^{s3}(\boldsymbol{k}) \{ [a_s^+(\boldsymbol{k}), a_3^-(\boldsymbol{k})]_+ - [a_s^-(\boldsymbol{k}), a_3^+(\boldsymbol{k})]_+ \} = 0 \tag{11.20a}$$

$$\sum_{s=1,2} \int \mathrm{d}^3\boldsymbol{k} l_{\mu\nu}^{s3}(\boldsymbol{k}) \{ [a_s^+(\boldsymbol{k}), a_3^-(\boldsymbol{k})]_+ - [a_s^-(\boldsymbol{k}), a_3^+(\boldsymbol{k})]_+ \} = 0, \tag{11.20b}$$

where $[A, B]_+ := A \circ B + B \circ A$ is the anticommutator of operators A and B. Since in (11.20) enter the anticommutators $[a_s^\pm(\boldsymbol{k}), a_3^\mp(\boldsymbol{k})]_+$ and $a_3^\mp(\boldsymbol{k})$ are actually free parameters, the contributions of $a_3^\pm(\boldsymbol{k})$ in the spin and orbital momentum operators can be eliminated via the following change in the theory. Redefine the normal products of creation and/or annihilation operators by assigning to them an additional (with respect to the definition in Sect. 9) multiplier (sign) equal to $(-1)^f$, where f is equal to the number of transpositions of the operators $a_3^\pm(\boldsymbol{k})$, relative to $a_1^\mp(\boldsymbol{k})$ and $a_2^\mp(\boldsymbol{k})$, required to be obtained the normal form of a product, i.e. all creation operators to be to the left of all annihilation ones. Evidently, such a (redefined) normal ordering procedure transforms (11.20) into identities and, consequently, after it all dynamical variables become independent of the operators $a_3^\pm(\boldsymbol{k})$. A similar result will be valid if the normal ordering procedure, as defined in Sect. (9), holds only for the operators $a_1^\pm(\boldsymbol{k})$ and $a_2^\pm(\boldsymbol{k})$ and the operators $a_3^\pm(\boldsymbol{k})$ anticommute with them,

$$[a_3^\mp(\boldsymbol{k}), a_s^\pm(\boldsymbol{k})]_+ = 0 \qquad \text{for } s = 1, 2. \tag{11.21}$$

Notice, if we put (cf. (11.10))

$$a_3^\pm(\boldsymbol{k}) = 0, \tag{11.22}$$

the definitions of vacuum, normal ordering procedure, and equations (11.21) (and hence (11.1) and (11.2)) will be satisfied. Besides, these choices will naturally exclude from the theory the 'longitudinal' photons, represented in our theory by the vector $a_3^+(\boldsymbol{k})(\mathcal{X}_0)$, which have identically vanishing dynamical characteristics.

The above discussion shows that the operators $a_3^\pm(\boldsymbol{k})$ can naturally be considered as fermi or bose operators that anticommute with $a_s^\pm(\boldsymbol{k})$, $s = 1, 2$, i.e. with anomalous commutation relations between both sets of operators [27, appendix F].[45] Besides, this agrees with the above-modified normal ordering procedure, which will be accepted below in the present section.

As a result of (11.19), the commutation relations (8.2) for an electromagnetic field (in Lorenz gauge) read[46]

$$[a_s^\pm(\boldsymbol{k}), a_t^\pm(\boldsymbol{k}')]_- = 0 \quad [a_s^\mp(\boldsymbol{k}), a_t^\pm(\boldsymbol{k}')]_- = \pm\delta_{st}\delta^3(\boldsymbol{k} - \boldsymbol{k}') \qquad \text{for } s, t = 1, 2 \tag{11.23}$$

and the operators of the dynamical variables, after the (redefined) normal ordering is performed, for this field are (see (11.20) and (9.3)–(9.8) with $m = 0$ and $\tau(\mathcal{U}) = 1$)

$$\mathcal{P}_\mu = \sum_{s=1,2} \int k_\mu|_{k_0=\sqrt{\boldsymbol{k}^2}} \, a_s^+(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) \, \mathrm{d}^3\boldsymbol{k} \tag{11.24}$$

[45] Since this excludes the operators $a_3^\pm(\boldsymbol{k})$ from all physically significant quantities, one is free to choose bilinear or other commutation relations between $a_3^\pm(\boldsymbol{k})$.

[46] According to our general considerations, the relations (11.23) are equivalent to the Maxwell(-Lorentz) equations, but written in terms of creation and annihilation operators.

$$\mathcal{Q} = 0 \tag{11.25}$$

$$\mathcal{L}_{\mu\nu} = \sum_{s=1,2} \int \mathrm{d}^3\boldsymbol{k} (x_\mu k_\nu - x_\nu k_\mu)|_{k_0=\sqrt{\boldsymbol{k}^2}}\, a_s^+(\boldsymbol{k}) \circ a_s^-(\boldsymbol{k}) + \mathrm{i}\hbar \sum_{s,s'=1,2} \int \mathrm{d}^3\boldsymbol{k}\, l_{\mu\nu}^{ss'}(\boldsymbol{k}) a_s^+(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k})$$

$$+ \mathrm{i}\hbar \frac{1}{2} \sum_{s=1,2} \int \mathrm{d}^3\boldsymbol{k} \Big\{ a_s^+(\boldsymbol{k}) \Big(k_\mu \overleftrightarrow{\frac{\partial}{\partial k^\nu}} - k_\nu \overleftrightarrow{\frac{\partial}{\partial k^\mu}} \Big) \circ a_s^-(\boldsymbol{k}) \Big\} \Big|_{k_0=\sqrt{\boldsymbol{k}^2}} \tag{11.26}$$

$$\mathcal{S}_{\mu\nu} = \mathrm{i}\hbar \sum_{s,s'=1,2} \int \mathrm{d}^3\boldsymbol{k} \sigma_{\mu\nu}^{ss'}(\boldsymbol{k}) a_s^+(\boldsymbol{k}) \circ a_{s'}^-(\boldsymbol{k}) \tag{11.27}$$

$$\boldsymbol{\mathcal{R}} = \boldsymbol{0} \tag{11.28}$$

$$\mathcal{S}^1 = \mathcal{S}^2 = 0 \quad \mathcal{S}^3 = \mathrm{i}\hbar \int \mathrm{d}^3\boldsymbol{k} \{ a_1^+(\boldsymbol{k}) \circ a_2^-(\boldsymbol{k}) - a_2^+(\boldsymbol{k}) \circ a_1^-(\boldsymbol{k}) \}, \tag{11.29}$$

where, for the derivation of (11.28) and (11.29), we have used (6.28), (4.23), (4.25), and that, formally, (11.27) corresponds to (9.6) with $a_3^\pm(\boldsymbol{k}) = 0$. Thus, the operators $a_3^\pm(\boldsymbol{k})$ do not enter in all of the ;dynamical variables.[47]

From (11.24)–(11.29), it is evident that the particles of an electromagnetic field, called photons, coincide with their antiparticles, which agrees with the general considerations in Sect. 10. Besides, the state vectors $b_1^+(\boldsymbol{k})(\mathcal{X}_0)$ and $b_2^+(\boldsymbol{k})(\mathcal{X}_0)$, with $b_s^\pm(\boldsymbol{k})$ given by (6.30), describe photons with 4-momentum $(\sqrt{\boldsymbol{k}^2}, \boldsymbol{k})$, zero charge, and vectors of spin $\boldsymbol{\mathcal{R}} = \boldsymbol{0}$ and $\boldsymbol{\mathcal{S}} = (0, 0, +\hbar)$ or $\boldsymbol{\mathcal{S}} = (0, 0, -\hbar)$ (see (10.14)), i.e. their spin vector $\boldsymbol{\mathcal{S}}$ is collinear with $\boldsymbol{k}$ with projection value $+\hbar$ or $-\hbar$, respectively, on its direction.

It is worth to be mentioned, the commutation relations (7.7)–(7.10) for an electromagnetic field take their 'ordinary' form, i.e.

$$[\mathcal{P}_\mu, \mathcal{P}_\nu]_{\_} = 0 \tag{11.30}$$

$$[\tilde{\mathcal{Q}}, \mathcal{P}_\mu]_{\_} = 0 \tag{11.31}$$

$$[\tilde{\mathcal{S}}_{\mu\nu}, \mathcal{P}_\lambda]_{\_} = 0 \tag{11.32}$$

$$[\tilde{\mathcal{L}}_{\mu\nu}, \mathcal{P}_\lambda]_{\_} = -\mathrm{i}\hbar \{ \eta_{\lambda\mu} \mathcal{P}_\nu - \eta_{\lambda\nu} \mathcal{P}_\mu \}, \tag{11.33}$$

as a consequence of which the dynamical variables in momentum picture are (see (7.18)–(7.20))

$$\mathcal{Q} = \tilde{\mathcal{Q}} \tag{11.34}$$

$$\mathcal{S}_{\mu\nu} = \tilde{\mathcal{S}}_{\mu\nu} \tag{11.35}$$

$$\mathcal{L}_{\mu\nu} = \tilde{\mathcal{L}}_{\mu\nu} + (x_\mu - x_{0\,\mu}) \mathcal{P}_\nu - (x_\nu - x_{0\,\nu}) \mathcal{P}_\mu. \tag{11.36}$$

At last, as we said above, the commutators $[A_\mu^\pm(\boldsymbol{k}), A_\nu^\pm(\boldsymbol{k})]_{\_}$ and $[A_\mu^\mp(\boldsymbol{k}), A_\nu^\pm(\boldsymbol{k})]_{\_}$ cannot be computed without knowing the explicit form of $[a_s^\pm(\boldsymbol{k}), a_3^\pm(\boldsymbol{k})]_{\_}$ and $[a_s^\mp(\boldsymbol{k}), a_3^\pm(\boldsymbol{k})]_{\_}$ for $s = 1, 2, 3$. For instance, the additional conditions (11.22) lead to (11.15) with $\mathcal{A}$ for $\mathcal{U}$ and $\varepsilon, \varepsilon' = \pm, \mp$.

The so-obtained quantization rules for electromagnetic field, i.e. equations (11.23) together with (11.22), coincide with the ones when it is quantized in *Coulomb* gauge [4, 15], in which is assumed

$$\mathcal{A}_0 = 0 \quad \sum_{a=1}^{3} [\mathcal{P}^a, \mathcal{A}_a]_{\_} = 0 \tag{11.37}$$

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[47] This situation should be compared with similar one in the Gupta-Bleuler formalism. In it the contribution of the 'time' ('scalar') and 'longitudinal' photons, the last corresponding to our states $a_3^\pm(\boldsymbol{k})(\mathcal{X}_0)$, is removed from the *average 4-momentum of the admissible states*, but, for example, the 'longitudinal' photons have a generally *non-vanishing* part in the vector of spin $\boldsymbol{\mathcal{S}}$ — see, e.g., [1, eq. (12.19)].

in momentum picture. This is not accidental as (11.37) is a special case of (3.18). In fact, by virtue of (4.7), it is equivalent to $k^a\, \mathcal{A}_a = 0$ (with $k^2 = k_0^2 - \boldsymbol{k}^2 = 0$) which is tantamount to

$$0 = (a_3^+(\boldsymbol{k}) + a_3^-(\boldsymbol{k}))(k^a v_a^3(\boldsymbol{k})\big|_{m=0}) = \begin{cases} a_3^+(\boldsymbol{k}) + a_3^-(\boldsymbol{k}) & \text{for } \boldsymbol{k} \neq \boldsymbol{0} \\ 0 & \text{for } \boldsymbol{k} = \boldsymbol{0} \end{cases}, \qquad (11.38)$$

due to (5.6), (5.7), (5.17) and (4.23)–(4.24). Therefore any choice of $a_3^\pm(\boldsymbol{k})$ such that $a_3^+(\boldsymbol{k}) + a_3^-(\boldsymbol{k}) = 0$ reduces the Lorenz gauge to the Coulomb one. However, the particular choice (11.22) completely reduces our quantization method to the one in Coulomb gauge, as a little more derailed comparison of the both methods reveals. We shall end this discussion with the remark that the choice (11.22) is external to the Lagrangian formalism and, of course, it is not the only possible one in that scheme.

12. On the choice of Lagrangian

Our previous exploration of free vector fields was based on the Lagrangian (see (3.1))

$$\tilde{\mathcal{L}}' = \tilde{\mathcal{L}} = \frac{m^2c^4}{1+\tau(\tilde{\mathcal{U}})} \tilde{\mathcal{U}}_\mu^\dagger \circ \tilde{\mathcal{U}}^\mu + \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} \{-(\partial_\mu \tilde{\mathcal{U}}_\nu^\dagger) \circ (\partial^\mu \tilde{\mathcal{U}}^\nu) + (\partial_\mu \tilde{\mathcal{U}}^{\mu\dagger}) \circ (\partial_\nu \tilde{\mathcal{U}}^\nu)\} \quad (12.1)$$

in Heisenberg picture. In it the field operators $\tilde{\mathcal{U}}_\mu$ and their Hermitian conjugate $\tilde{\mathcal{U}}_\mu^\dagger$ do not enter on equal footing: in a sense, $\tilde{\mathcal{U}}_\mu^\dagger$ are 'first' and $\tilde{\mathcal{U}}_\mu$ are 'second' in order (counting from left to right) unless the field is neutral/Hermitian. Since $\tilde{\mathcal{U}}_\mu$ and $\tilde{\mathcal{U}}_\mu^\dagger$ are associated with the operators $a_s^\pm$ and $a_s^{\dagger\,\pm}$ (see Sect. 5), which create/annihilate field's particles and antiparticles, respectively, the Lagrangian (12.1) describes the particles and antiparticles in a non-symmetric way, which is non-desirable for a free field as for it what should be called a particle or antiparticle is more a convention than a natural distinction. This situation is usually corrected via an additional condition in the theory, such as the charge symmetry, spin-statistics theorem etc. Its sense is the inclusion in the theory of the symmetry particle $\leftrightarrow$ antiparticle, which in terms of the creation and annihilation operators should be expressed via theory's invariance under the change $a_s^\pm(\boldsymbol{k}) \leftrightarrow a_s^{\dagger\,\pm}(\boldsymbol{k})$. As we demonstrated in [13, 14] for free scalar and spin $\frac{1}{2}$ fields, this symmetry/invariance can be incorporated in the initial Lagrangian, from which the theory is constructed. Below we shall show how this can be achieved for free vector fields satisfying the Lorenz condition.

As an alternative to the Lagrangian (12.1), one can consider

$$\tilde{\mathcal{L}}'' = \frac{m^2c^4}{1+\tau(\tilde{\mathcal{U}})} \tilde{\mathcal{U}}_\mu \circ \tilde{\mathcal{U}}^{\mu\dagger} + \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} \{-(\partial_\mu \tilde{\mathcal{U}}_\nu) \circ (\partial^\mu \tilde{\mathcal{U}}^{\nu\dagger}) + (\partial_\mu \tilde{\mathcal{U}}^\mu) \circ (\partial_\nu \tilde{\mathcal{U}}^{\nu\dagger})\} = \tilde{\mathcal{L}}'\big|_{\tilde{\mathcal{U}}_\mu \leftrightarrow \tilde{\mathcal{U}}_\mu^\dagger} \quad (12.2)$$

in which the places, where the operators $\tilde{\mathcal{U}}_\mu$ and $\tilde{\mathcal{U}}_\mu^\dagger$ are situated in (12.1), are interchanged. In terms of particles and antiparticles, this means that we call fields particles antiparticles and *vice versa*, or, equivalently, that the change $a_s^\pm(\boldsymbol{k}) \leftrightarrow a_s^{\dagger\,\pm}(\boldsymbol{k})$ has been made. Obviously, the Lagrangian (12.2) suffers from the same problems as (12.1). However, judging by our experience in [13] and partially in [14], we can expect that the half-sum of the Lagrangians (12.1) and (12.2), i.e.

$$\begin{aligned}\tilde{\mathcal{L}}''' = \frac{m^2c^4}{2(1+\tau(\tilde{\mathcal{U}}))} \{\tilde{\mathcal{U}}_\mu^\dagger \circ \tilde{\mathcal{U}}^\mu + \tilde{\mathcal{U}}_\mu \circ \tilde{\mathcal{U}}^{\mu\dagger}\} + \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} \{&-(\partial_\mu \tilde{\mathcal{U}}_\nu^\dagger) \circ (\partial^\mu \tilde{\mathcal{U}}^\nu) + (\partial_\mu \tilde{\mathcal{U}}^{\mu\dagger}) \circ (\partial_\nu \tilde{\mathcal{U}}^\nu) \\ &- (\partial_\mu \tilde{\mathcal{U}}_\nu) \circ (\partial^\mu \tilde{\mathcal{U}}^{\nu\dagger}) + (\partial_\mu \tilde{\mathcal{U}}^\mu) \circ (\partial_\nu \tilde{\mathcal{U}}^{\nu\dagger})\} = \frac{1}{2}(\tilde{\mathcal{L}}' + \tilde{\mathcal{L}}''), \quad (12.3)\end{aligned}$$

is one of the Lagrangians we are looking for, as it is invariant under the change $\tilde{\mathcal{U}}_\mu \leftrightarrow \tilde{\mathcal{U}}^\dagger_\mu$.

To any one of the Lagrangians (12.1)–(12.3), we add the Lorenz conditions

$$\partial^\mu \tilde{\mathcal{U}}_\mu = 0 \quad \partial^\mu \tilde{\mathcal{U}}^\dagger_\mu = 0, \tag{12.4}$$

which are symmetric under the transformation $\tilde{\mathcal{U}}_\mu \leftrightarrow \tilde{\mathcal{U}}^\dagger_\mu$ and for $m = 0$ are additional conditions for the Lagrangian formalism, but for $m \neq 0$ they are consequences from the field equations (see Sect. 3 and below).

According to the general rules of Sect. 2 (see (2.4) and (2.16)), the Lagrangians (12.1)–(12.3) and the Lorenz conditions (12.4) in momentum picture respectively are:

$$\mathcal{L}' = \frac{m^2c^4}{1+\tau(\mathcal{U})} \mathcal{U}^\dagger_\mu \circ \mathcal{U}^\mu + \frac{c^2}{1+\tau(\mathcal{U})} \{ [\mathcal{U}^\dagger_\nu, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^\nu, \mathcal{P}^\mu]_{\_} - [\mathcal{U}^{\mu\dagger}, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^\nu, \mathcal{P}_\nu]_{\_} \} \tag{12.5}$$

$$\mathcal{L}'' = \frac{m^2c^4}{1+\tau(\mathcal{U})} \mathcal{U}_\mu \circ \mathcal{U}^{\mu\dagger} + \frac{c^2}{1+\tau(\mathcal{U})} \{ [\mathcal{U}_\nu, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^{\nu\dagger}, \mathcal{P}^\mu]_{\_} - [\mathcal{U}^{\mu}, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^{\nu\dagger}, \mathcal{P}_\nu]_{\_} \} \tag{12.6}$$

$$\begin{aligned}\mathcal{L}''' = \frac{m^2c^4}{2(1+\tau(\mathcal{U}))} \{ \mathcal{U}^\dagger_\mu \circ \mathcal{U}^\mu + \mathcal{U}_\mu \circ \mathcal{U}^{\mu\dagger} \} + \frac{c^2}{2(1+\tau(\mathcal{U}))} \{ [\mathcal{U}^\dagger_\nu, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^\nu, \mathcal{P}^\mu]_{\_} \\ - [\mathcal{U}^{\mu\dagger}, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^\nu, \mathcal{P}_\nu]_{\_} + [\mathcal{U}_\nu, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^{\nu\dagger}, \mathcal{P}^\mu]_{\_} - [\mathcal{U}^{\mu}, \mathcal{P}_\mu]_{\_} \circ [\mathcal{U}^{\nu\dagger}, \mathcal{P}_\nu]_{\_} \}\end{aligned} \tag{12.7}$$

$$[\mathcal{U}_\mu, \mathcal{P}^\mu]_{\_} = 0 \quad [\mathcal{U}^\dagger_\mu, \mathcal{P}^\mu]_{\_} = 0. \tag{12.8}$$

The derivatives of the above Lagrangians happen to coincide and are as follows:[48]

$$\begin{aligned}&\frac{\partial \mathcal{L}'}{\partial \mathcal{U}^\mu} = \frac{\partial \mathcal{L}''}{\partial \mathcal{U}^\mu} = \frac{\partial \mathcal{L}'''}{\partial \mathcal{U}^\mu} = m^2c^4 \mathcal{U}^\dagger_\mu \quad \frac{\partial \mathcal{L}'}{\partial \mathcal{U}^{\mu\dagger}} = \frac{\partial \mathcal{L}''}{\partial \mathcal{U}^{\mu\dagger}} = \frac{\partial \mathcal{L}'''}{\partial \mathcal{U}^{\mu\dagger}} = m^2c^4 \mathcal{U}_\mu \\ &\pi_{\mu\lambda} = \frac{\partial \mathcal{L}'}{\partial y^{\mu\lambda}} = \frac{\partial \mathcal{L}''}{\partial y^{\mu\lambda}} = \frac{\partial \mathcal{L}'''}{\partial y^{\mu\lambda}} = \mathrm{i}\hbar c^2 [\mathcal{U}^\dagger_\mu, \mathcal{P}_\lambda]_{\_} - \mathrm{i}\hbar c^2 \eta_{\mu\lambda} [\mathcal{U}^{\nu\dagger}, \mathcal{P}_\nu]_{\_} \\ &\pi^\dagger_{\mu\lambda} = \frac{\partial \mathcal{L}'}{\partial y^{\mu\lambda\dagger}} = \frac{\partial \mathcal{L}''}{\partial y^{\mu\lambda\dagger}} = \frac{\partial \mathcal{L}'''}{\partial y^{\mu\lambda\dagger}} = \mathrm{i}\hbar c^2 [\mathcal{U}_\mu, \mathcal{P}_\lambda]_{\_} - \mathrm{i}\hbar c^2 \eta_{\mu\lambda} [\mathcal{U}^{\nu}, \mathcal{P}_\nu]_{\_}\end{aligned} \tag{12.9}$$

with $y_{\mu\lambda} := \frac{1}{\mathrm{i}\hbar}[\mathcal{U}_\mu, \mathcal{P}_\lambda]_{\_}$ and $y^\dagger_{\mu\lambda} := \frac{1}{\mathrm{i}\hbar}[\mathcal{U}^\dagger_\mu, \mathcal{P}_\lambda]_{\_}$.

As a consequence of (12.9), the field equations of the Lagrangians (12.5)–(12.7) coincide and are given by (3.13) (and (12.8) as an additional conditions/equation for $m = 0$) which, for $m \neq 0$, split into the Klein-Gordon equations (3.17) and the Lorenz conditions (12.8). From here it follows that the material of sections 4 and 5 remains valid without any changes for the Lagrangian theories arising from any one of the Lagrangians (12.5)–(12.7) (under the Lorenz conditions in the massless case).

The densities of the operators of the dynamical variables for the Lagrangian (12.1) are given via (3.22)–(3.26). Similarly, the energy-momentum tensor, (charge) current and spin angular momentum operators for the Lagrangians (12.2) and (12.3) respectively are:[49]

$$\begin{aligned}\tilde{\mathcal{T}}''_{\mu\nu} &= \frac{1}{1+\tau(\tilde{\mathcal{U}})} \{ (\partial_\nu \tilde{\mathcal{U}}^\lambda) \circ \tilde{\pi}_{\lambda\mu} + \tilde{\pi}^\dagger_{\lambda\mu} \circ (\partial_\nu \tilde{\mathcal{U}}^{\lambda\dagger}) \} - \eta_{\mu\nu} \tilde{\mathcal{L}}'' \\ &= -\frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})} \{ (\partial_\mu \tilde{\mathcal{U}}_\lambda) \circ (\partial_\nu \tilde{\mathcal{U}}^{\lambda\dagger}) + (\partial_\nu \tilde{\mathcal{U}}_\lambda) \circ (\partial_\mu \tilde{\mathcal{U}}^{\lambda\dagger}) \} - \eta_{\mu\nu} \tilde{\mathcal{L}}''\end{aligned} \tag{12.10a}$$

[48] This assertion is valid if the derivatives are calculated according to the classical rules of analysis of commuting variables, as it is done below. Such an approach requires additional rules for ordering of the operators entering into the expressions for the dynamical variables, as the ones presented below. Both of these assumptions, in the particular cases we are considering here, have their rigorous explanation in a different way for computing derivatives of *non*-commuting variables, as it is demonstrated in [17], to which paper the reader is referred for further details.

[49] Excluding the spin operators, the other density operators can be obtained, by virtue of (12.4), as sums of similar ones corresponding to $\mathcal{U}_0$, $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ and considered as free scalar fields — see [13]. For a rigorous derivation, see [17].

$$\tilde{\mathcal{J}}''_\mu = \frac{q}{\mathrm{i}\hbar}\{\tilde{\mathcal{U}}^\lambda\circ\tilde{\pi}_{\lambda\mu} - \tilde{\pi}^\dagger_{\lambda\mu}\circ\tilde{\mathcal{U}}^{\lambda\dagger}\} = \mathrm{i}\hbar q c^2\{-(\partial_\mu\tilde{\mathcal{U}}_\lambda)\circ\tilde{\mathcal{U}}^{\lambda\dagger} + \tilde{\mathcal{U}}_\lambda\circ(\partial_\mu\tilde{\mathcal{U}}^{\lambda\dagger})\} \tag{12.10b}$$

$$\begin{aligned}\tilde{\mathcal{S}}''^{\lambda}_{\mu\nu} &:= \frac{1}{1+\tau(\tilde{\mathcal{U}})}\{(I^\sigma_{\rho\mu\nu}\tilde{\mathcal{U}}_\sigma)\circ\tilde{\pi}^{\rho\lambda} + \tilde{\pi}^{\rho\lambda\dagger}\circ(I^{\dagger\sigma}_{\rho\mu\nu}\tilde{\mathcal{U}}^\dagger_\sigma)\}\\ &= \frac{\hbar^2c^2}{1+\tau(\tilde{\mathcal{U}})}\{(\partial^\lambda\tilde{\mathcal{U}}_\mu)\circ\tilde{\mathcal{U}}^\dagger_\nu - (\partial^\lambda\tilde{\mathcal{U}}_\nu)\circ\tilde{\mathcal{U}}^\dagger_\mu - \tilde{\mathcal{U}}_\mu\circ(\partial^\lambda\tilde{\mathcal{U}}^\dagger_\nu) + \tilde{\mathcal{U}}_\mu\circ(\partial^\lambda\tilde{\mathcal{U}}^\dagger_\mu)\}\end{aligned} \tag{12.10c}$$

$$\begin{aligned}\tilde{\mathcal{T}}'''_{\mu\nu} = \frac{1}{2(1+\tau(\tilde{\mathcal{U}}))}\{&\tilde{\pi}_{\lambda\mu}\circ(\partial_\nu\tilde{\mathcal{U}}^\lambda) + (\partial_\nu\tilde{\mathcal{U}}^{\lambda\dagger})\circ\tilde{\pi}^\dagger_{\lambda\mu}\\ &+ (\partial_\nu\tilde{\mathcal{U}}^\lambda)\circ\tilde{\pi}_{\lambda\mu} + \tilde{\pi}^\dagger_{\lambda\mu}\circ(\partial_\nu\tilde{\mathcal{U}}^{\lambda\dagger})\} - \eta_{\mu\nu}\tilde{\mathcal{L}}''' = \frac{1}{2}\{\tilde{\mathcal{T}}'_{\mu\nu} + \tilde{\mathcal{T}}''_{\mu\nu}\}\end{aligned} \tag{12.11a}$$

$$\tilde{\mathcal{J}}'''_\mu = \frac{q}{2\mathrm{i}\hbar}\{\tilde{\pi}_{\lambda\mu}\circ\tilde{\mathcal{U}}^\lambda - \tilde{\mathcal{U}}^{\lambda\dagger}\circ\tilde{\pi}^\dagger_{\lambda\mu} + \tilde{\mathcal{U}}^\lambda\circ\tilde{\pi}_{\lambda\mu} - \tilde{\pi}^\dagger_{\lambda\mu}\circ\tilde{\mathcal{U}}^{\lambda\dagger}\} = \frac{1}{2}\{\tilde{\mathcal{J}}'_\mu + \tilde{\mathcal{J}}''_\mu\} \tag{12.11b}$$

$$\begin{aligned}\tilde{\mathcal{S}}'''^{\lambda}_{\mu\nu} := \frac{1}{2(1+\tau(\tilde{\mathcal{U}}))}\{&\tilde{\pi}^{\rho\lambda}\circ(I^\sigma_{\rho\mu\nu}\tilde{\mathcal{U}}_\sigma) + (I^{\dagger\sigma}_{\rho\mu\nu}\tilde{\mathcal{U}}^\dagger_\sigma)\circ\tilde{\pi}^{\rho\lambda\dagger}\\ &+ (I^\sigma_{\rho\mu\nu}\tilde{\mathcal{U}}_\sigma)\circ\tilde{\pi}^{\rho\lambda} + \tilde{\pi}^{\rho\lambda\dagger}\circ(I^{\dagger\sigma}_{\rho\mu\nu}\tilde{\mathcal{U}}^\dagger_\sigma)\} = \frac{1}{2}\{\tilde{\mathcal{S}}'^{\lambda}_{\mu\nu} + \tilde{\mathcal{S}}''^{\lambda}_{\mu\nu}\}.\end{aligned} \tag{12.11c}$$

Thus, we see that the dynamical variables derived from $\mathcal{L}''$ can be obtained from the ones for $\mathcal{L}' = \mathcal{L}$ by making the change $\mathcal{U}_\mu \leftrightarrow \mathcal{U}^\dagger_\mu$ and reversing the current's sign. Besides, the dynamical variables for $\mathcal{L}'''$ are equal to the half-sum of the corresponding ones for $\mathcal{L}' = \mathcal{L}$ and $\mathcal{L}''$. So, symbolically we can write (see also (2.1), (2.19)–(2.21) and (2.4))

$$\mathcal{D}'' = \pm\mathcal{D}'\big|_{\mathcal{U}_\mu\leftrightarrow\mathcal{U}^\dagger_\mu} \qquad \mathcal{D}''' = \frac{1}{2}(\mathcal{D}' + \mathcal{D}''), \tag{12.12}$$

where $\mathcal{D} = \mathcal{T}_{\mu\nu}, \mathcal{J}_\mu, \mathcal{S}^\lambda_{\mu\nu}, \mathcal{L}^\lambda_{\mu\nu}, \mathcal{P}_\mu, \mathcal{Q}, \mathcal{S}_{\mu\nu}, \mathcal{L}_{\mu\nu}$ and the minus sign in the first equality stand only for $\mathcal{D} = \mathcal{J}_\mu, \mathcal{Q}$. If we express the dynamical variables in terms of creation and annihilation operators, which are identical for the Lagrangians we consider (*vide infra*), then (12.12) takes the form

$$\mathcal{D}'' = \pm\mathcal{D}'\big|_{a^\pm_s(\boldsymbol{k})\leftrightarrow a^{\dagger\,\pm}_s(\boldsymbol{k})} \qquad \mathcal{D}''' = \frac{1}{2}(\mathcal{D}' + \mathcal{D}''), \tag{12.13}$$

with $\mathcal{D} = \mathcal{P}_\mu, \mathcal{Q}, \mathcal{S}_{\mu\nu}, \mathcal{L}_{\mu\nu}$.[50] To save some space, we shall write explicitly only the conserved operator quantities for the Lagrangian (12.6). Combining (6.5)–(6.7) and (6.15) with the rule (12.13), we get in Heisenberg picture (and before normal ordering):

$$\mathcal{P}''_\mu = \frac{1}{1+\tau(\mathcal{U})}\sum_{s=1}^{3-\delta_{0m}}\int k_\mu\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\{a^+_s(\boldsymbol{k})\circ a^{\dagger\,-}_s(\boldsymbol{k}) + a^-_s(\boldsymbol{k})\circ a^{\dagger\,+}_s(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{12.14a}$$

$$\tilde{\mathcal{Q}}'' = -q\sum_{s=1}^{3-\delta_{0m}}\int\{a^+_s(\boldsymbol{k})\circ a^{\dagger\,-}_s(\boldsymbol{k}) - a^-_s(\boldsymbol{k})\circ a^{\dagger\,+}_s(\boldsymbol{k})\}\,\mathrm{d}^3\boldsymbol{k} \tag{12.14b}$$

$$\tilde{\mathcal{S}}''_{\mu\nu} = \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\sum_{s,s'=1}^{3}\int\mathrm{d}^3\boldsymbol{k}\,\sigma^{ss'}_{\mu\nu}(\boldsymbol{k})\{a^+_s(\boldsymbol{k})\circ a^{\dagger\,-}_{s'}(\boldsymbol{k}) - a^-_s(\boldsymbol{k})\circ a^{\dagger\,+}_{s'}(\boldsymbol{k})\} \tag{12.14c}$$

$$\begin{aligned}\tilde{\mathcal{L}}''_{\mu\nu} = \frac{1}{1+\tau(\mathcal{U})}&\sum_{s=1}^{3-\delta_{0m}}\int\mathrm{d}^3\boldsymbol{k}(x_{0\,\mu}k_\nu - x_{0\,\nu}k_\mu)\big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}\{a^+_s(\boldsymbol{k})\circ a^{\dagger\,-}_s(\boldsymbol{k}) + a^-_s(\boldsymbol{k})\circ a^{\dagger\,+}_s(\boldsymbol{k})\}\\ &+ \frac{\mathrm{i}\hbar}{1+\tau(\mathcal{U})}\sum_{s,s'=1}^{3}\int\mathrm{d}^3\boldsymbol{k}\,l^{ss'}_{\mu\nu}(\boldsymbol{k})\{a^+_s(\boldsymbol{k})\circ a^{\dagger\,-}_{s'}(\boldsymbol{k}) - a^-_s(\boldsymbol{k})\circ a^{\dagger\,+}_{s'}(\boldsymbol{k})\}\end{aligned}$$

[50] For $\mathcal{D} = \mathcal{P}_\mu, \mathcal{Q}$, the first equation in (12.13) is evident (see (6.1) and (6.2)), but for $\mathcal{D} = \mathcal{S}_{\mu\nu}, \mathcal{L}_{\mu\nu}$ some simple manipulations are required for its proof — see (6.8) and (6.16).

$$
+\frac{\mathrm{i}\hbar}{2(1+\tau(\mathcal{U}))}\sum_{s=1}^{3-\delta_{0m}}\int \mathrm{d}^3\boldsymbol{k}\Big\{a_s^{+}(\boldsymbol{k})\Big(k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}-k_\nu\overleftrightarrow{\frac{\partial}{\partial k^\mu}}\Big)\circ a_s^{\dagger\,-}(\boldsymbol{k})
$$

$$
-a_s^{-}(\boldsymbol{k})\Big(k_\mu\overleftrightarrow{\frac{\partial}{\partial k^\nu}}-k_\nu\overleftrightarrow{\frac{\partial}{\partial k^\mu}}\Big)\circ a_s^{\dagger\,+}(\boldsymbol{k})\Big\}\Big|_{k_0=\sqrt{m^2c^2+\boldsymbol{k}^2}}, \qquad (12.14\text{d})
$$

Let us turn now our attention to the field equations in terms of creation and annihilation operators for the Lagrangians $\mathcal{L}'$, $\mathcal{L}''$ and $\mathcal{L}'''$. For $\mathcal{L}'=\mathcal{L}$ they are given by (7.4). To derive them for $\mathcal{L}''$ and $\mathcal{L}'''$, one should repeat the derivation of (7.4) from (7.2) with $\mathcal{P}''_\mu$ and $\mathcal{P}'''_\mu=\frac{1}{2}(\mathcal{P}'_\mu+\mathcal{P}''_\mu)$, respectively, for $\mathcal{P}_\mu$. In this way, from (7.2) with $\mathcal{P}_\mu=\mathcal{P}'_\mu,\mathcal{P}''_\mu$, (12.14a) and (12.13) with $\mathcal{D}=\mathcal{P}_\mu$, we obtain the field equations derived from the Lagrangians $\mathcal{L}''$ and $\mathcal{L}'''$ respectively as:

$$
\big[a_s^{\pm}(\boldsymbol{k}),a_t^{+}(\boldsymbol{q})\circ a_t^{\dagger\,-}(\boldsymbol{q})+a_t^{-}(\boldsymbol{q})\circ a_t^{\dagger\,+}(\boldsymbol{q})\big]_{\_}\pm(1+\tau(\mathcal{U}))a_s^{\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})={}''f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q}) \qquad (12.15\text{a})
$$

$$
\big[a_s^{\dagger\,\pm}(\boldsymbol{k}),a_t^{+}(\boldsymbol{q})\circ a_t^{\dagger\,-}(\boldsymbol{q})+a_t^{-}(\boldsymbol{q})\circ a_t^{\dagger\,+}(\boldsymbol{q})\big]_{\_}\pm(1+\tau(\mathcal{U}))a_s^{\dagger\,\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})={}''f_{st}^{\dagger\,\pm}(\boldsymbol{k},\boldsymbol{q}) \qquad (12.15\text{b})
$$

$$
\big[a_s^{\pm}(\boldsymbol{k}),a_t^{\dagger\,+}(\boldsymbol{q})\circ a_t^{-}(\boldsymbol{q})+a_t^{\dagger\,-}(\boldsymbol{q})\circ a_t^{+}(\boldsymbol{q})\big]_{\_}+\big[a_s^{\pm}(\boldsymbol{k}),a_t^{+}(\boldsymbol{q})\circ a_t^{\dagger\,-}(\boldsymbol{q})+a_t^{-}(\boldsymbol{q})\circ a_t^{\dagger\,+}(\boldsymbol{q})\big]_{\_}
$$

$$
\pm 2(1+\tau(\mathcal{U}))a_s^{\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})={}'''f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q}) \qquad (12.16\text{a})
$$

$$
\big[a_s^{\dagger\,\pm}(\boldsymbol{k}),a_t^{\dagger\,+}(\boldsymbol{q})\circ a_t^{-}(\boldsymbol{q})+a_t^{\dagger\,-}(\boldsymbol{q})\circ a_t^{+}(\boldsymbol{q})\big]_{\_}+\big[a_s^{\dagger\,\pm}(\boldsymbol{k}),a_t^{+}(\boldsymbol{q})\circ a_t^{\dagger\,-}(\boldsymbol{q})+a_t^{-}(\boldsymbol{q})\circ a_t^{\dagger\,+}(\boldsymbol{q})\big]_{\_}
$$

$$
\pm 2(1+\tau(\mathcal{U}))a_s^{\dagger\,\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q})={}'''f_{st}^{\dagger\,\pm}(\boldsymbol{k},\boldsymbol{q}), \qquad (12.16\text{b})
$$

where the polarization indices s and t take the values

$$
s,t=\begin{cases}1,2,3 & \text{for } m\neq 0\\ 1,2 & \text{for } m=0\end{cases} \qquad (12.17)
$$

and the operator-valued (generalized) functions ${}^{a}f^{\pm}(\boldsymbol{k},\boldsymbol{q})$ and ${}^{a}f^{\dagger\,\pm}(\boldsymbol{k},\boldsymbol{q})$, with $a={}',{}'',{}'''$, are such that

$$
\int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}}\,{}^{a}f^{\pm}(\boldsymbol{k},\boldsymbol{q})\,\mathrm{d}^3\boldsymbol{q}=\int q_\mu\big|_{q_0=\sqrt{m^2c^2+\boldsymbol{q}^2}}\,{}^{a}f^{\dagger\,\pm}(\boldsymbol{k},\boldsymbol{q})\,\mathrm{d}^3\boldsymbol{q}=0. \qquad (12.18)
$$

Equations, similar to (7.7)–(7.10), can be derived form (12.15) and (12.16) and, consequently, expressions for the dynamical variables in momentum picture, similar to (7.18)–(7.20), can easily be obtained from these equations.

As we see, the dynamical variables and the field equations in terms of creation and annihilation operators for the Lagrangians (12.1)–(12.3) are completely different for a non-Hermitian field, $\mathcal{U}_\mu^\dagger\neq\mathcal{U}_\mu$ or $a_s^{\dagger\,\pm}(\boldsymbol{k})\neq a_s^{\pm}(\boldsymbol{k})$, and, in this sense, the arising from them quantum field theories of free vector field satisfying the Lorenz condition are different. A step toward the identification of these theories is achieved via the 'second' quantization procedure, i.e. by establishing/imposing for/on the creation and annihilation operators commutation relations, like (8.2) for the Lagrangian (12.1). These relations for the Lagrangians (12.2) and (12.3) can be derived analogously to the ones for (12.1), i.e. by making appropriate changes in the derivation of the commutation relations for an arbitrary free scalar field, given in [13] (see also [14], where free spinor fields are investigates). Without going into details, we shall

say that this procedure results into the commutation relations (8.2) for any one of the Lagrangians (12.1)–(12.3) (under the Lorenz conditions (12.4) in the massless case). In this way, the systems of field equations (7.4), (12.15) and (12.16) became identical and equivalent to (8.2).

It should be emphasized, the derivation of (8.2) for the Lagrangians (12.1)–(12.3) is not identical [13]: the Lagrangian (12.3) does not admit quantization via *anti*commutators, contrary to (12.1) and (12.2). So, the establishment of (8.2) for $\mathcal{L}'$ and $\mathcal{L}''$ requires as an additional hypothesis the quantization via commutators or some equivalent to it assertion, like the charge symmetry, spin-statistics theorem, etc. [1]. Said differently, this additional assumption is not needed for the Lagrangian (12.3) as it entails such a hypothesis in the framework of the Lagrangian formalism. The initial cause for this state of affairs is that the symmetry particle $\leftrightarrow$ antiparticle is encoded in the Lagrangian $\mathcal{L}'''$ via its invariance under the change $\mathcal{U}_\mu \leftrightarrow \mathcal{U}_\mu^\dagger$. In particular, since for a neutral field we, evidently, have

$$\mathcal{L}' = \mathcal{L}'' = \mathcal{L}''' \qquad \text{if } \mathcal{U}_\mu = \mathcal{U}_\mu^\dagger, \tag{12.19}$$

for such a field, e.g. for the electromagnetic one, the spin-statistics theorem and other equivalent to it assertions are consequences form the Lagrangian formalism investigated in the present paper.

Since the commutation relations for the Lagrangians (12.1)–(12.3) are identical, we assume the normal ordering procedures and the definitions of the vacuum for them to be identical, respectively, and to coincide with the ones given in Sect. 9.

Applying the normal ordering procedure to the dynamical variables corresponding to the Lagrangians (12.1)–(12.3) (see (6.5)–(6.8), (6.15), (12.14) and (12.13)), we see that, after this operation, they became independent of the Lagrangian we have started, i.e. symbolically we can write

$$\mathcal{D}' = \mathcal{D}'' = \mathcal{D}''' = \mathcal{D} \qquad \mathcal{D} = \mathcal{P}_\mu,\, \mathcal{Q},\, \mathcal{L}_{\mu\nu},\, \mathcal{S}_{\mu\nu}, \tag{12.20}$$

where the operators for $\mathcal{L}' = \mathcal{L}$ are given by (9.3)–(9.6). (To prove these equations for $\mathcal{D} = \mathcal{L}_{\mu\nu}, \mathcal{S}_{\mu\nu}$, one has to use the antisymmetry of the quantities (6.8) and (6.16).)

Let us summarize at the end. The Lagrangians (12.1)–(12.3), which are essentially different for non-Hermitian fields, generally entail quite different Lagrangian field theories unless some additional conditions are added to the Lagrangian formalism. In particular, these theories became identical if one assumes the commutation relations (8.2), the normal ordering procedure and the definition of vacuum, as given in Sect. 9. The Lagrangian (12.3) has the advantage that the spin-statistics theorem (or charge symmetry, etc.) is encoded in it, while, for the Lagrangians (12.1) and (12.2) this assertion should be postulated (imposed) as an additional condition to the Lagrangian formalism. For a neutral free vector field satisfying the Lorenz condition, i.e. for electromagnetic field in Lorenz gauge, the spin-statistics theorem is a consequence from the Lagrangian formalism.[51]

13. On the role of the Lorenz condition in the massless case

Until now, in the description of free massless vector fields, we supposed that they satisfy the Lorenz condition, i.e. the equations $\partial^\mu \tilde{\mathcal{U}}_\mu = 0$ and $\partial^\mu \tilde{\mathcal{U}}_\mu^\dagger = 0$ in Heisenberg picture or (3.18) in momentum one, as subsidiary restrictions to the Lagrangian formalism. Such a theory contains some physical, not mathematical, problems which were summarized and

[51] Recall [13], the proof of the spin-statistics theorem (charge symmetry, etc.) for the Lagrangian (12.3) requires as a hypothesis, additional to the Lagrangian formalism, the assertion that the commutators or anticommutators of all combinations of creation and/or annihilation operators to be proportional to the identity mapping of system's Hilbert space of states, i.e. to be c-numbers.

partially analyzed in Sect. 11. The present section is devoted to a brief exploration of a Lagrangian formalism for free massless vector field without additional restrictions, like the Lorenz condition. As we shall see, in this case the problems inherent to a formalism with the Lorenz conditions remain and new ones are added to them.

For other point of view on the topic of this section, see, e.g., [4, § 7.1].

Most of the considerations in this section will be done in Heisenberg picture which will prevent the exposition from new details (which are not quite suitable for the purpose).

13.1. Description of free massless vector fields (without the Lorenz condition)

The description of a free massless vector field coincides with the one of a free massive vector field, given in Sect. 3, with the only difference that the field's mass parameter m (which is equal to the mass of field's particles, if $m \neq 0$) is set equal to zero,

$$m = 0. \tag{13.1}$$

In particular, the Lagrangian formalism can start from the Lagrangian (see (3.1) and (3.7))

$$\tilde{\mathcal{L}} = -\frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})}(\partial_\mu \tilde{\mathcal{U}}_\nu^\dagger) \circ (\partial^\mu \tilde{\mathcal{U}}^\nu) + \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})}(\partial_\mu \tilde{\mathcal{U}}^{\mu\dagger}) \circ (\partial_\nu \tilde{\mathcal{U}}^\nu) \tag{13.2}$$

$$\mathcal{L} = \frac{c^2}{1+\tau(\mathcal{U})}\{[\mathcal{U}_\nu^\dagger, \mathcal{P}_\mu]_\_ \circ [\mathcal{U}^\nu, \mathcal{P}^\mu]_\_ - [\mathcal{U}^{\mu\dagger}, \mathcal{P}_\mu]_\_ \circ [\mathcal{U}^\nu, \mathcal{P}_\nu]_\_\} \tag{13.3}$$

in Heisenberg and momentum picture, respectively. The Euler-Lagrange equations for this Lagrangian are the following massless Proca equations (see (3.5) and (3.13))

$$\tilde{\Box}(\tilde{\mathcal{U}}_\mu) - \partial_\mu(\partial^\lambda \tilde{\mathcal{U}}_\lambda) = 0 \qquad \tilde{\Box}(\tilde{\mathcal{U}}_\mu^\dagger) - \partial_\mu(\partial^\lambda \tilde{\mathcal{U}}_\lambda^\dagger) = 0 \tag{13.4}$$

$$[[\mathcal{U}_\mu, \mathcal{P}_\lambda]_\_, \mathcal{P}^\lambda]_\_ - [[\mathcal{U}_\nu, \mathcal{P}^\nu]_\_, \mathcal{P}^\mu]_\_ = 0 \qquad [[\mathcal{U}_\mu^\dagger, \mathcal{P}_\lambda]_\_, \mathcal{P}^\lambda]_\_ - [[\mathcal{U}_\nu^\dagger, \mathcal{P}^\nu]_\_, \mathcal{P}^\mu]_\_ = 0 \tag{13.5}$$

in Heisenberg and momentum picture, respectively. (Recall, $\tilde{\Box} := \partial^\mu\partial_\mu$.) As pointed in Sect. 3, these equations do not imply that the field operators satisfy the massless Klein-Gordon equations and the Lorenz conditions (see (3.6), (3.13), (3.17) and (3.18) with $m = 0$). The common solutions of the massless Klein-Gordon equations and the Lorenz conditions for the field operators are solutions of the massless Proca equation, but the opposite is not necessary, i.e. the latter system of equations is more general than the former one. This is the cause why, for solutions of (3.5), the Lagrangian (13.3) cannot be reduced to (3.19) with $m = 0$ in the general case (unless the Lorenz conditions (3.18) are imposed on the solutions of (13.5) as additional conditions).

The general expressions for the densities of the dynamical variables through the generalize momenta $\tilde{\pi}_{\lambda\mu}$ (see the first equalities in (3.22)–(3.26)) remain, of course, valid in the massless case too, but their particular dependence on the field operators is different from the second equalities in (3.22)–(3.26), as now (3.12), without the Lorenz conditions on these operators, should be used. Thus, the dynamical variables of a massless vector field are:

$$\begin{aligned}\tilde{\mathcal{T}}_{\mu\nu} = &-\frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})}\{(\partial_\mu \tilde{\mathcal{U}}_\lambda^\dagger) \circ (\partial_\nu \tilde{\mathcal{U}}^\lambda) + (\partial_\nu \tilde{\mathcal{U}}_\lambda^\dagger) \circ (\partial_\mu \tilde{\mathcal{U}}^\lambda)\} - \eta_{\mu\nu}\,\tilde{\mathcal{L}}\\ &+ \frac{c^2\hbar^2}{1+\tau(\tilde{\mathcal{U}})}\{(\partial_\varkappa \tilde{\mathcal{U}}^{\varkappa\dagger}) \circ (\partial_\nu \tilde{\mathcal{U}}_\mu) + (\partial_\nu \tilde{\mathcal{U}}_\mu^\dagger) \circ (\partial_\varkappa \tilde{\mathcal{U}}^\varkappa)\}\end{aligned} \tag{13.6}$$

$$\tilde{\mathcal{J}}_\mu = \mathrm{i}\hbar q c^2\{(\partial_\mu \tilde{\mathcal{U}}_\lambda^\dagger) \circ \tilde{\mathcal{U}}^\lambda - \tilde{\mathcal{U}}_\lambda^\dagger \circ (\partial_\mu \tilde{\mathcal{U}}^\lambda) - (\partial_\varkappa \tilde{\mathcal{U}}^{\varkappa\dagger}) \circ \tilde{\mathcal{U}}_\mu + \tilde{\mathcal{U}}_\mu^\dagger \circ (\partial_\varkappa \tilde{\mathcal{U}}^\varkappa)\} \tag{13.7}$$

$$\tilde{\mathcal{M}}_{\mu\nu}^\lambda = \tilde{\mathcal{L}}_{\mu\nu}^\lambda + \tilde{\mathcal{S}}_{\mu\nu}^\lambda \tag{13.8}$$

$$\tilde{\mathcal{L}}_{\mu\nu}^\lambda := x_\mu\,\tilde{\mathcal{T}}^\lambda{}_\nu - x_\nu\,\tilde{\mathcal{T}}^\lambda{}_\mu \tag{13.9}$$

$$
\begin{aligned}
\tilde{\mathcal{S}}^{\lambda}_{\mu\nu} = \frac{\hbar^2 c^2}{1+\tau(\tilde{\mathcal{U}})} \{ & (\partial^\lambda \tilde{\mathcal{U}}^\dagger_\mu) \circ \tilde{\mathcal{U}}_\nu - (\partial^\lambda \tilde{\mathcal{U}}^\dagger_\nu) \circ \tilde{\mathcal{U}}_\mu - \tilde{\mathcal{U}}^\dagger_\mu \circ (\partial^\lambda \tilde{\mathcal{U}}_\nu) + \tilde{\mathcal{U}}^\dagger_\nu \circ (\partial^\lambda \tilde{\mathcal{U}}_\mu) \\
& - \delta^\lambda_\mu \big((\partial_\varkappa \tilde{\mathcal{U}}^{\varkappa\dagger}) \circ \tilde{\mathcal{U}}_\nu + \tilde{\mathcal{U}}^\dagger_\nu \circ (\partial_\varkappa \tilde{\mathcal{U}}^{\varkappa}) \big) + \delta^\lambda_\nu \big((\partial_\varkappa \tilde{\mathcal{U}}^{\varkappa\dagger}) \circ \tilde{\mathcal{U}}_\mu + \tilde{\mathcal{U}}^\dagger_\mu (\partial_\varkappa \tilde{\mathcal{U}}^{\varkappa}) \big) \}.
\end{aligned} \tag{13.10}
$$

Notice, in (13.6) the Lagrangian $\tilde{\mathcal{L}}$ must be replaced by its value given by (13.2), not by (3.19) with $m = 0$. Evidently, the Lorenz conditions (3.6b) reduce the equations (13.6)–(13.10) to (3.22)–(3.26), respectively. If needed, the reader can easily write the above equations in momentum picture by means of the general rules of Sect. 2.

It should be remarked, as the energy-momentum tensor (13.6) is non-symmetric, the spin and orbital angular momentum are no longer conserved quantities.

13.2. Analysis of the Euler-Lagrange equations

Since the solutions of the Euler-Lagrange equations (13.4) (or (13.5)) generally do not satisfy the Klein-Gordon equation, we cannot apply to free massless vector fields the methods developed for free scalar fields. To explore the equations (13.4), we shall transform them into algebraic ones in the *momentum representation* in Heisenberg picture [1, 15, 16].

Define the Fourier images $\tilde{u}_\mu(k)$ and $\tilde{u}^\dagger_\mu(k)$, $k \in \mathbb{R}^4$, of the field operators via the Fourier transforms

$$
\tilde{\mathcal{U}}_\mu(x) = \frac{1}{(2\pi)^2} \int \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar} kx} \tilde{u}_\mu(k) \, \mathrm{d}^3 k \qquad \tilde{\mathcal{U}}^\dagger_\mu(x) = \frac{1}{(2\pi)^2} \int \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar} kx} \tilde{u}^\dagger_\mu(k) \, \mathrm{d}^3 k, \tag{13.11}
$$

where $\mathrm{d}^4 k := \mathrm{d}k^0 \, \mathrm{d}k^1 \, \mathrm{d}k^2 \, \mathrm{d}k^3$ and $kx := k_\mu x^\mu$. Since $\tilde{\mathcal{U}}^\dagger_\mu(x)$ is the Hermitian conjugate of $\tilde{\mathcal{U}}_\mu(x)$, $\tilde{\mathcal{U}}^\dagger_\mu(x) := (\tilde{\mathcal{U}}_\mu(x))^\dagger$, we have

$$
\tilde{u}^\dagger_\mu(k) = (\tilde{u}_\mu(-k))^\dagger. \tag{13.12}
$$

Substituting (13.11) into (13.4), we find the systems of equations

$$
k^2 \tilde{u}_\mu(k) - k_\mu k^\nu \tilde{u}_\nu(k) = 0 \qquad k^2 \tilde{u}^\dagger_\mu(k) - k_\mu k^\nu \tilde{u}^\dagger_\nu(k) = 0, \tag{13.13}
$$

which is equivalent to (13.4). Here and below $k^2 := k_\mu k^\mu$. (From the context, it will be clear that, in most cases, by k^2 we have in mind $k_\mu k^\mu$, not the second contravariant component of k.)

Let us consider the classical analogue of the equation (13.13), i.e.

$$
k^2 v_\mu(k) - k_\mu k^\nu v_\nu(k) \equiv (k^2 \eta_{\mu\nu} - k_\mu k_\nu) v^\nu(k) = 0, \tag{13.14}
$$

where $v_\mu(k)$ is a classical, not operator-valued, vector field over the k-space $\mathbb{R}^4$. This is a linear homogeneous system of 4 equations for the 4 variables $v_0(k)$, $v_1(k)$, $v_2(k)$ and $v_3(k)$. Since the determinant of the matrix of (13.14) is[52]

$$
\det[k^2 \eta_{\mu\nu} - k_\mu k_\nu]^3_{\mu,\nu=0} = k^2(-k^2 + k^2) \equiv 0
$$

the system of equation (13.14) possesses always a non-zero solution relative to $v^\nu(k)$. Besides, the form of this determinant indicates that the value $k^2 = 0$ is crucial for the number of linearly independent solutions of (13.14). A simple algebraic calculation reveals that the rank r of the matrix $[k^2 \eta_{\mu\nu} - k_\mu k_\nu]^3_{\mu,\nu=0}$, as a function of k, is: $r = 0$ if $k_\mu = 0$, $r = 1$ if

[52] One can easily prove that $\det\big[c_{\mu\nu} - z_\mu z_\nu\big]^3_{\mu,\nu=0} = c_0 c_1 c_2 c_3 + z_0^2 c_1 c_2 c_3 + c_0 z_1^2 c_2 c_3 + c_0 c_1 z_2^2 c_3 + c_0 c_1 c_2 z_3^2$ for a diagonal matrix $[c_{\mu\nu}] = \operatorname{diag}(c_0, c_2, c_2, c_3)$ and any 4-vector z_μ. Putting here $c_{\mu\nu} = k^2 \eta_{\mu\nu}$ and $z_\mu = k_\mu$, we get the cited result.

$k^2 = 0$ and $k_\mu \neq 0$ for some $\mu = 0, 1, 2, 3$ and $r = 3$ if $k^2 \neq 0$. Respectively, the number of linearly independent solutions of (13.14) is infinity if $k_\mu = 0$, three if $k^2 = 0$ and $k_\mu \neq 0$ for some $\mu = 0, 1, 2, 3$, and one if $k^2 \neq 0$.

For $k^2 = 0$, the system (13.14) reduces to the equation (4.20) with $m = 0$, which was investigated in Sect. 4. For $k^2 \neq 0$, it has the evident solution

$$w_\mu(k) := \mathrm{i}\frac{k_\mu}{\sqrt{k^2}} \qquad (\, k^2 \neq 0\,), \tag{13.15}$$

which is normalized to -1,

$$w_\mu(k) w^\nu(k) = -1 \qquad (\, k^2 \neq 0\,), \tag{13.16}$$

and any other solution of (13.14) is proportional to $w_\mu(k)$, as defined by (13.15).

Therefore, for any k_μ, we can write the general solution of (13.14) as

$$v_\mu(k) = \delta_{0k^2} \sum_{s=1}^{3} \alpha_s(k) v_\mu^s(\boldsymbol{k})\big|_{m=0} + (1 - \delta_{0k^2}) \alpha_4(k) w_\mu(k), \tag{13.17}$$

where $\alpha_1(k), \dots, \alpha_4(k)$ are some functions of $k = (k_0, \dots, k_3)$, $v_\mu^s(\boldsymbol{k})$ with $s = 1, 2, 3$ are defined by (4.20)–(4.25), and the Kronecker delta-symbol δ_{0k^2} ($:= 1$ for $k^2 = 0$ and $:= 0$ for $k^2 \neq 0$) takes care of the number of linearly independent solutions of (13.14).

Returning to the operator equations (13.13), we can express their solutions as

$$\begin{aligned}
\tilde{u}_\mu(k) &= (2\pi)^2 \{\mathrm{i}c^2 (2\pi\hbar)^3\}^{-1/2} \{\delta_{0k^2} \sum_{s=1}^{3} \tilde{a}_s(k) v_\mu^s(\boldsymbol{k})\big|_{m=0} + (1 - \delta_{0k^2}) \tilde{a}_4(k) w_\mu(k)\} \\
\tilde{u}_\mu^\dagger(k) &= (2\pi)^2 \{\mathrm{i}c^2 (2\pi\hbar)^3\}^{-1/2} \{\delta_{0k^2} \sum_{s=1}^{3} \tilde{a}_s^\dagger(k) v_\mu^s(\boldsymbol{k})\big|_{m=0} + (1 - \delta_{0k^2}) \tilde{a}_4^\dagger(k) w_\mu(k)\},
\end{aligned} \tag{13.18}$$

where $\tilde{a}_1(k), \dots, \tilde{a}_4^\dagger(k)$ are some operator-valued functions of k, which, by (13.12), are such that

$$\tilde{a}_\omega^\dagger(k) = (\tilde{a}_\omega(-k))^\dagger \qquad \omega = 1, 2, 3, 4, \tag{13.19}$$

and the factor $(2\pi)^2 \{\mathrm{i}c^2 (2\pi\hbar)^3\}^{-1/2}$ is introduced for future convenience. (The operators $\tilde{a}_1(k), \dots, \tilde{a}_4^\dagger(k)$ are closely related to the creation and annihilation operators, but we shall not consider this problem here.)

At last, combining (13.19) and (13.11), we can write the solutions of the field equations (13.4) as

$$\begin{aligned}
\tilde{\mathcal{U}}_\mu(x) =& (2\pi)^2 \{\mathrm{i}c^2 (2\pi\hbar)^3\}^{-1/2} \int\limits_{k^2=0} \sum_{s=1}^{3} \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}kx} \tilde{a}_s(k) v_\mu^s(\boldsymbol{k})\big|_{m=0} \, \mathrm{d}^4 k \\
&+ (2\pi)^2 \{\mathrm{i}c^2 (2\pi\hbar)^3\}^{-1/2} \int\limits_{k^2\neq 0} \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}kx} \tilde{a}_4(k) w_\mu(\boldsymbol{k}) \, \mathrm{d}^4 k \\
\tilde{\mathcal{U}}_\mu^\dagger(x) =& (2\pi)^2 \{\mathrm{i}c^2 (2\pi\hbar)^3\}^{-1/2} \int\limits_{k^2=0} \sum_{s=1}^{3} \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}kx} \tilde{a}_s^\dagger(k) v_\mu^s(\boldsymbol{k})\big|_{m=0} \, \mathrm{d}^4 k \\
&+ (2\pi)^2 \{\mathrm{i}c^2 (2\pi\hbar)^3\}^{-1/2} \int\limits_{k^2\neq 0} \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}kx} \tilde{a}_4^\dagger(k) w_\mu(\boldsymbol{k}) \, \mathrm{d}^4 k.
\end{aligned} \tag{13.20}$$

Since the Lorenz conditions (see (3.6))

$$\partial^\mu \tilde{\mathcal{U}}_\mu(x) = 0 \qquad \partial^\mu \tilde{\mathcal{U}}_\mu^\dagger(x) = 0 \tag{13.21}$$

in momentum representation in Heisenberg picture read (see (13.11)),

$$k^\mu \tilde{u}_\mu(k) = 0 \qquad k^\mu \tilde{u}_\mu^\dagger(k) = 0, \tag{13.22}$$

we see that they are equivalent to the selection of solutions of (13.13) with

$$k^2 = 0. \tag{13.23}$$

Said differently, the Lorenz condition on the field operators is equivalent to the imposition of the restrictions

$$\tilde{a}_4(k) = 0 \qquad \tilde{a}_4^\dagger(k) = 0 \qquad (\,k^2 \neq 0\,), \tag{13.24}$$

due to (13.18) (or (13.20)). Here the operators $\tilde{a}_4(k)$ and $\tilde{a}_4^\dagger(k)$ may be considered as a measure of the satisfaction of the Lorenz condition by the field operators. Therefore the sum of the terms containing an integral over the hyperboloid $k^2 = 0$ in (13.20) corresponds to field operators satisfying the Lorenz condition and, consequently, to them is valid the theory developed in the preceding sections. In particular, up to a constant, the operators $\tilde{a}_s(k)$, $s = 1, 2, 3$, are sums of the creation and annihilation operators (in Heisenberg picture — see (6.36)) of a free massless vector field satisfying the Lorenz condition.

13.3. Dynamical variables

To reveal the meaning of the operators $\tilde{a}_4(k)$ and $\tilde{a}_4^\dagger(k)$ in (13.20), we shall express the field's dynamical variables in terms of $\tilde{a}_\omega(k)$ and $\tilde{a}_\omega^\dagger(k)$, $\omega = 1, 2, 3, 4$. For the purpose, the decompositions (13.20) should be inserted into the expressions (13.6)–(13.10) and, then, the conserved operators (2.1) , (2.19)–(2.21) to be calculated. It is not difficult to be seen, a dynamical variable $\tilde{\mathcal{D}}$, with $\mathcal{D} = \mathcal{P}_\mu,\ \mathcal{Q},\ \mathcal{L}_{\mu\nu},\ \mathcal{S}_{\mu\nu}$ for respectively the momentum, charge, orbital and spin angular momentum operators, has the following structure:

$$\tilde{\mathcal{D}} = {}^{0}\tilde{\mathcal{D}} + {}^{0-4}\tilde{\mathcal{D}} + {}^{4}\tilde{\mathcal{D}}. \tag{13.25}$$

Here ${}^{0}\tilde{\mathcal{D}} := \int_{k^2=0} \mathrm{d}^4k \int_{k'^2=0} \mathrm{d}^4k' \{\cdots\}$ is the dynamical variable under the conditions (13.24), i.e. if the field operators were supposed to satisfy the Lorenz condition; the second term is of the form ${}^{0-4}\tilde{\mathcal{D}} := \int_{k^2=0} \mathrm{d}^4k \int_{k'^2\neq0} \mathrm{d}^4k' \{\cdots\}$ with the expression in braces being a linear combination of terms like $\tilde{a}_s^\dagger(k) \circ \tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k') \circ \tilde{a}_s(k)$, where $s = 1, 2, 3$; and the structure of the last term is ${}^{4}\tilde{\mathcal{D}} := \int_{k^2\neq0} \mathrm{d}^4k \int_{k'^2\neq0} \mathrm{d}^4k' \{\cdots\}$ with the expression in braces being proportional to the operator $\tilde{a}_4^\dagger(k) \circ \tilde{a}_4(k')$.

By means of the explicit formulae (4.23)–(4.25) and (13.15), one can prove, after simple algebraic calculations, that

$${}^{4}\tilde{\mathcal{D}} = 0 \qquad \tilde{\mathcal{D}} = \tilde{\mathcal{P}}_\mu,\ \tilde{\mathcal{Q}},\ \tilde{\mathcal{L}}_{\mu\nu},\ \tilde{\mathcal{S}}_{\mu\nu}. \tag{13.26}$$

Thus, if we regard $\tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k')$ as independent degrees of freedom (possibly connected with some particles), then their pure ('free') contribution to the dynamical variables is vanishing. However, the second term in (13.25) is generally non-zero. Simple, but long and tedious, algebraic calculations give the following results:

$$\begin{aligned} {}^{0-4}\tilde{\mathcal{P}}_\mu = \frac{1}{1+\tau(\tilde{\mathcal{U}})} \sum_{s=1}^{3} \int_{k^2=0} \mathrm{d}^4k \int_{k'^2\neq0} \mathrm{d}^4k' \delta^3(\boldsymbol{k}+\boldsymbol{k}') \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}(k_0-k_0')x^0} \\ \times \frac{k_0 k_\mu'}{\sqrt{{k_0'}^2 - k_0^2}} \Big\{ \sqrt{k_0^2}\,\delta^{3s} - k_0 \Big\} \{ \tilde{a}_s^\dagger(k) \circ \tilde{a}_4(k') + \tilde{a}_4^\dagger(k') \circ \tilde{a}_s(k) \} \end{aligned} \tag{13.27a}$$

$$ {}^{0-4}\tilde{\mathcal{Q}} = q \sum_{s=1}^{3} \int_{k^2=0} \mathrm{d}^4 k \int_{k'^2\neq 0} \mathrm{d}^4 k' \delta^3(\boldsymbol{k}+\boldsymbol{k}') \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}(k_0-k'_0)x^0} $$
$$ \times \frac{k_0}{\sqrt{{k'_0}^2 - k_0^2}} \Big\{ \sqrt{k_0^2}\delta^{3s} - k_0 \Big\} \{ -\tilde{a}_s^\dagger(k)\circ\tilde{a}_4(k') + \tilde{a}_4^\dagger(k')\circ\tilde{a}_s(k) \} \quad (13.27b) $$
$$ {}^{0-4}\tilde{\mathcal{S}}_{\mu\nu} = \frac{\mathrm{i}\hbar}{1+\tau(\tilde{\mathcal{U}})} \sum_{s=1}^{3} \int_{k^2=0} \mathrm{d}^4 k \int_{k'^2\neq 0} \mathrm{d}^4 k' \delta^3(\boldsymbol{k}+\boldsymbol{k}') \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}(k_0-k'_0)x^0} $$
$$ \times \frac{1}{\sqrt{{k'_0}^2 - k_0^2}} \{ \tilde{a}_s^\dagger(k)\circ\tilde{a}_4(k') + \tilde{a}_4^\dagger(k')\circ\tilde{a}_s(k) \} $$
$$ \times \begin{cases} (-1)^{\delta_{0\mu}} \Big\{ k_0(k'_0-k_0)v_a^s(\boldsymbol{k}) - k'_a\big(\sqrt{k_0^2}\delta^{3s} - k'_0\big) \Big\} & \text{for } (\mu,\nu)=(0,a),(a,0) \text{ with } a=1,2,3 \\ (k_0-k'_0)(k_\mu v_\nu^s(\boldsymbol{k}) - k_\nu v_\mu^s(\boldsymbol{k})) & \text{otherwise}. \end{cases} $$
(13.27c)

Notice, the expression ${k'_0}^2 - k_0^2$ in (13.27) is different from zero as ${k'_0}^2 - k_0^2 = {k'_0}^2 - \boldsymbol{k}^2 = {k'_0}^2 - \boldsymbol{k}'^2 = k'^2 \neq 0$, due to $0 = k^2 = k_0^2 - \boldsymbol{k}^2$ and the δ-function $\delta^3(\boldsymbol{k}+\boldsymbol{k}')$ in (13.27).

13.4. The field equations

Recall now that we consider quantum field theories in which the Heisenberg relations (2.7) hold as a subsidiary restriction on the field operators. Consequently, the system of field equations consists of the Euler-Lagrange equations (13.4), the Heisenberg relations

$$ [\tilde{\mathcal{U}}_\mu(x), \mathcal{P}_\nu]_{\_} = \mathrm{i}\hbar \frac{\partial \tilde{\mathcal{U}}_\mu(x)}{\partial x^\nu} \qquad [\tilde{\mathcal{U}}_\mu^\dagger(x), \mathcal{P}_\nu]_{\_} = \mathrm{i}\hbar \frac{\partial \tilde{\mathcal{U}}_\mu^\dagger(x)}{\partial x^\nu} \quad (13.28) $$

and the explicit connection between $\mathcal{P}_\mu$ and the field operators, i.e. (see (13.25)–(13.27))

$$ \tilde{\mathcal{P}}_\mu = {}^{0}\tilde{\mathcal{P}}_\mu + {}^{0-4}\tilde{\mathcal{P}}_\mu \quad (13.29) $$

with ${}^{0}\tilde{\mathcal{P}}_\mu$ given by the r.h.s. of (6.5) and ${}^{0-4}\tilde{\mathcal{P}}_\mu$ defined via (13.27a).

Since the expansions (13.20) take care of the Euler-Lagrange equation (13.4), the equations (13.28) remain the only restrictions on the field operators. Substituting equation (13.20) into (13.28), we get

$$ [\tilde{a}_s(k), \mathcal{P}_\mu]_{\_} = -k_\mu \tilde{a}_s(k) \qquad s=1,2 \quad k^2=0 \quad (13.30a) $$
$$ [\tilde{a}_4(k), \mathcal{P}_\mu]_{\_} = -k_\mu \tilde{a}_4(k) \qquad k^2 \neq 0. \quad (13.30b) $$

One can verify that (13.30a) is equivalent to (7.2) with $m=0$ and $\mathcal{P}_\mu$ given by (13.29). Notice, the operators $\tilde{a}_3(k)$ and $\tilde{a}_3^\dagger(k)$ (or $a_3^\pm(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$), with $k^2=0$, enter in the field equations (13.30) and into the dynamical variables via (13.29) and (13.27) (see also (6.7) and (6.15)). However, a particle interpretation of the degrees of freedom connected with $\tilde{a}_3(k)$ and $\tilde{a}_3^\dagger(k)$ fails as they enter in (13.30), (6.7) and (6.15) only in combinations/compositions with $\tilde{a}_\omega(k)$ and $\tilde{a}_\omega^\dagger(k)$ with $\omega = 1,2,4$. In that sense, the operators $\tilde{a}_3(k)$ and $\tilde{a}_3^\dagger(k)$ serve as 'coupling constants' with respect to the remaining ones. Similar is the situation with the operators $\tilde{a}_4(k)$ and $\tilde{a}_4^\dagger(k)$, with $k^2 \neq 0$, regarding the dynamical variables, but these operators are 'more dynamical' as they must satisfy the equations (13.30b).

The explicit equations of motion for $\tilde{a}_\omega(k)$ and $\tilde{a}_\omega^\dagger(k)$, $\omega = 1,2,3,4$, can be obtained by inserting (13.29) (see also (13.27a) and (6.5) with $m=0$) into (13.30). The result will be similar to (7.3) or (7.4), with $m=0$, but additional terms, depending on $\tilde{a}_4(k)$ and $\tilde{a}_4^\dagger(k)$ with $k^2 \neq 0$, will be presented. We shall not write these equations as they will not be used further and it seems that $\tilde{a}_4(k) = \tilde{a}_4^\dagger(k) = 0$, when they coincide with (7.4), is the only their physically meaningful solution (see Subsect. 13.5 below).

13.5. Discussion

Equations (13.26) and (13.27) show that the operators $\tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k')$ (with $k'^2 \neq 0$) do not have their own contributions to the dynamical variables, but they do contribute to them via the combinations $\tilde{a}_s^\dagger(k) \circ \tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k') \circ \tilde{a}_s(k)$, with $k^2 = 0$, $k'^2 \neq 0$ and $s = 1, 2, 3$. In a sense, the operators $\tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k')$ act as 'operator-valued coupling constants' for the operators $\tilde{a}_s^\dagger(k)$ and $\tilde{a}_s(k)$ (and hence for $\tilde{a}_s^{\dagger\pm}(k)$ and $\tilde{a}_s^\pm(k)$), as via them they bring an additional contribution to the dynamical variables with respect to vector fields satisfying the Lorenz condition. In this aspect, the operators $\tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k')$ are similar to $a_3^\pm(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$ (or $\tilde{a}_3(k)$ and $\tilde{a}_3^\dagger(k)$) (for details, see Sect. 11).

Consequently, the absence of the Lorenz conditions brings new problems, in addition to the similar ones with the operators $a_3^\pm(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$ (see Sect. 11). One can say that a massless free vector field, which does not satisfy the Lorenz condition, is equivalent to a similar field satisfying that restriction and with self-interaction determined by $\tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k')$. One can try to get rid of the contributions of the last operators in the dynamical variables by imposing on them some subsidiary conditions, called gauges in terms of the initial operators $\tilde{\mathcal{U}}_\mu(x)$ and $\tilde{\mathcal{U}}_\mu^\dagger(x)$. The problem of gauge freedom of massless vector fields is well-studied in the literature [1, 2, 4] to which the reader is referred. In particular, one can set the operators (13.27) to zero by demanding

$$\tilde{a}_s^\dagger(k) \circ \tilde{a}_4(k') + \tilde{a}_4^\dagger(k') \circ \tilde{a}_s(k) = 0 \quad \tilde{a}_s^\dagger(k) \circ \tilde{a}_4(k') - \tilde{a}_4^\dagger(k') \circ \tilde{a}_s(k) = 0 \tag{13.31}$$

for $k^2 = 0$, $k'^2 \neq 0$, and $s = 1, 2, 3$, or, equivalently,

$$\tilde{a}_s^\dagger(k) \circ \tilde{a}_4(k') = 0 \quad \tilde{a}_4^\dagger(k') \circ \tilde{a}_s(k) = 0 \qquad (\,k^2 = 0,\ k'^2 \neq 0,\ s = 1, 2, 3\,). \tag{13.32}$$

For instance, these equalities are identically satisfied if the Lorenz condition, e.g. in the form (13.24), is valid.

Let us summarize. A theory of massless free vector field, based on the Lagrangian (13.2) contains as free parameters the operators $a_3^\pm(\boldsymbol{k})$ and $a_3^{\dagger\pm}(\boldsymbol{k})$ and, partially, $\tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k')$, with $k^2 = 0$ and $k'^2 \neq 0$. These operators have, generally, non-vanishing contributions to the dynamical variables only via their compositions with the ('physical') operators $a_s^\pm(\boldsymbol{k})$ and $a_s^{\dagger\pm}(\boldsymbol{k})$ (or $\tilde{a}_s(k)$ and $\tilde{a}_s^\dagger(k)$), with $s = 1, 2$, and via the combinations $\tilde{a}_3^\dagger(k) \circ \tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k') \circ \tilde{a}_3(k)$. As a result, these operators describe degrees of freedom with vanishing own dynamical characteristics and, consequently, they do not admit particle interpretation.[53] As we saw in Sect. 11, the operators $a_s^\pm(\boldsymbol{k})$ $a_s^{\dagger\pm}(\boldsymbol{k})$, with $s = 1, 2, 3$ and $k^2 \neq 0$, describe reasonably well massless free vector fields satisfying the Lorenz condition, in particular the electromagnetic field. From this point of view, we can say that the operators $a_3^\pm(\boldsymbol{k})$, $a_3^{\dagger\pm}(\boldsymbol{k})$, $\tilde{a}_4(k')$ and $\tilde{a}_4^\dagger(k')$ describe some selfinteraction of the preceding field, but it seems such a selfinteracting, massless, free vector field is not known to exist in the Nature at the moment. The easiest way for exclusion of that selfinteraction from the theory is the pointed operators to be set equal to zero, i.e. on the field operators to be imposed the Lorenz conditions and (11.10). However, it is possible that other restrictions on the Lagrangian formalism may achieve the same goal.

At the end, the above considerations point that the Lorenz condition should be imposed as an addition (subsidiary) condition to the Lagrangian formalism of massless free vector fields, in particular to the quantum theory of free electromagnetic field. Besides, the conditions (11.10) also seems to be necessary for a satisfactory description of these fields.

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[53] If one assigns particle interpretation of the discussed operators, then they will have vanishing 4-momentum, charge and spin and hence will be unobservable.

14. Conclusion

A more or less detailed Lagrangian quantum field theory of free vector fields, massless in Lorenz gauge and massive ones, in momentum picture was constructed in the present paper. Regardless of a common treatment of the both types of fields, the massless case has some specific features and problems. The Lorenz conditions are external to the Lagrangian formalism of massless vector fields, but they are compatible with it. However, for an electromagnetic field, which is a neutral massless vector field, in Lorenz gauge, we have obtained a problem-free description in terms of creation and annihilation operators, i.e. in terms of particles. This description is similar to the Gupta-Bleuler quantization of electromagnetic field, but is quite different from the latter one and it is free of the problems this formalism contains. Our formalism reproduces, under suitable additional conditions, the quantization of electromagnetic field in Coulomb gauge.

Between the Lagrangians, considered for a suitable description of free vector fields satisfying the Lorenz conditions, we have singled out the Lagrangian (12.3). It is invariant under the transformation particle↔antiparticle, described in appropriate variables, so that in it is encoded the charge symmetry (or spin-statistics theorem). The field equations in terms of creation and annihilation operators for this Lagrangian are (12.16) (under the conditions (12.17) and (12.18)). They can equivalently be rewritten as

$$\begin{aligned}\big[[a_t^{\dagger\,+}(\boldsymbol{q}),a_t^-(\boldsymbol{q})]_+,a_s^{\pm}(\boldsymbol{k})\big]_- &+ \big[[a_t^+(\boldsymbol{q}),a_t^{\dagger\,-}(\boldsymbol{q})]_+,a_s^{\pm}(\boldsymbol{k})\big]_- \\ &= \pm 2(1+\tau(\mathcal{U}))a_s^{\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q}) - {}'''f_{st}^{\pm}(\boldsymbol{k},\boldsymbol{q})\end{aligned} \tag{14.1a}$$

$$\begin{aligned}\big[[a_t^{\dagger\,+}(\boldsymbol{q}),a_t^-(\boldsymbol{q})]_+,a_s^{\dagger\,\pm}(\boldsymbol{k})\big]_- &+ \big[[a_t^+(\boldsymbol{q}),a_t^{\dagger\,-}(\boldsymbol{q})]_+,a_s^{\dagger\,\pm}(\boldsymbol{k})\big]_- \\ &= \pm 2(1+\tau(\mathcal{U}))a_s^{\dagger\,\pm}(\boldsymbol{k})\delta_{st}\delta^3(\boldsymbol{k}-\boldsymbol{q}) - {}'''f_{st}^{\dagger\,\pm}(\boldsymbol{k},\boldsymbol{q}).\end{aligned} \tag{14.1b}$$

Trilinear equations of this kind are typical for the so-called parastatistics and parafield theory [28–32], in which they play a role of (para)commutation relations. In a forthcoming paper, we intend to demonstrate how the parabose commutation relations for free vector fields (satisfying the Lorenz condition) can be obtained from (14.1).

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